Unconditional Metric Approximation Property and subspaces of L^1.

Joint work with G. Godefroy and N. Kalton

X separable Banach space.

Metric Approximation Property:

$(\forall \varepsilon > 0)$ $\quad \exists R_n : X \to X \quad rk(R_n) < \infty$

(1) $R_n x \xrightarrow[n\to\infty]{} x, \ \forall x \in X$

(2) $\|R_n\| \leqslant 1+\varepsilon \ , \ \forall n \geqslant 1$.

Equivalently:

(1') $x = \sum_{n=1}^{\infty} S_n x \ , \ \forall x \in X$

(2') $\sup_{N \geqslant 1} \left\| \sum_{n=1}^{N} S_n \right\| \leqslant 1+\varepsilon$.

$(S_n = R_n - R_{n-1} \ , \ R_0 = 0)$.

Unconditional Metric Approximation Property (UMAP).

$\forall \varepsilon > 0 \ , \ \exists T_n : X \to X \quad rk(T_n) < +\infty$

(1') $x = \sum_{n=1}^{\infty} T_n x \ , \ \forall x \in X$

(3) $\sup_{N \geqslant 1} \sup_{\varepsilon_n = \pm 1} \left\| \sum_{n=1}^{N} \varepsilon_n T_n \right\| \leqslant 1+\varepsilon$.

Pełczyński - Wojtaszczyk (1976) : X has (UMAP) iff : $\forall \varepsilon > 0 \quad \exists Z_\varepsilon \supset X$ with $(1+\varepsilon)$-unconditional FDD and $\underset{\text{projection}}{\exists P_\varepsilon} : Z_\varepsilon \to X$
$\|P_\varepsilon\| \leqslant 1+\varepsilon$.

Casazza - Kalton (1990) : X has (UMAP) iff : $\exists R_n : X \to X \quad rk(R_n) < +\infty$

(1) $R_n x \xrightarrow[n\to\infty]{} x$, $\forall x \in X$

(2") $\lim\limits_{n\to\infty} \|I - 2R_n\| = 1$.

Intrinsic characterization → geometrical properties.

Detailed study by
G. Godefroy - N. Kalton - P. Saphar
(Studia Math. 1993).
In particular:

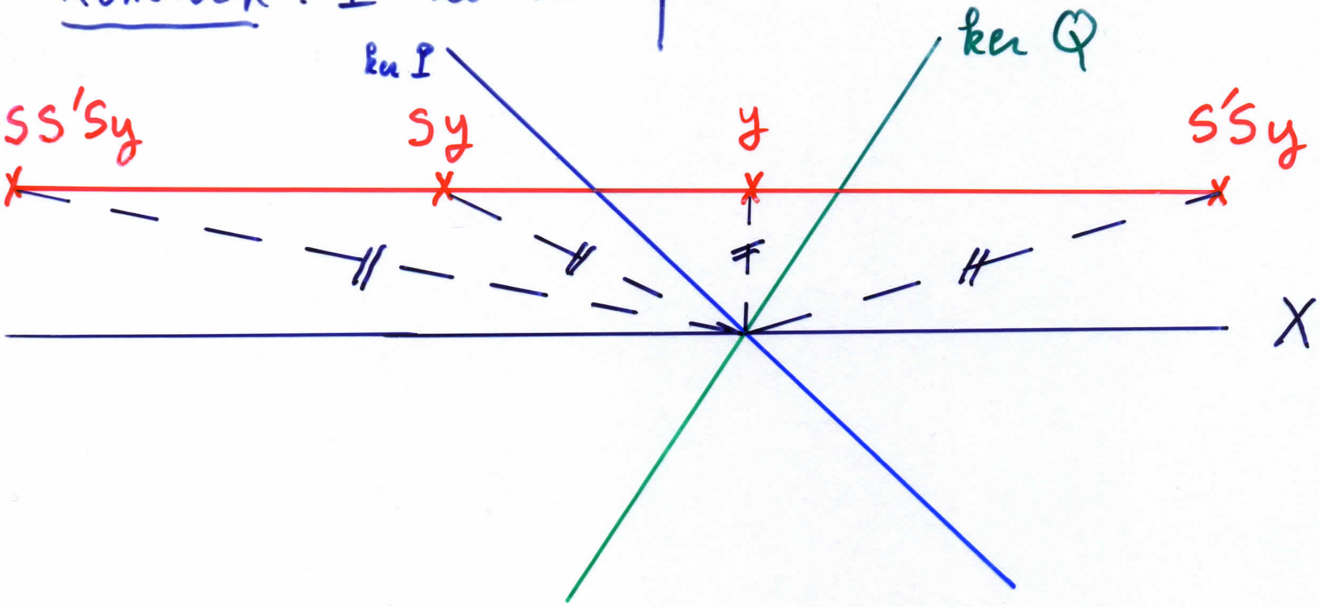
Proposition 1. If X has (UMAP) and $X \not\supset c_0$, then $\underset{\text{projection}}{\exists P} : X^{**} \to X$, with
$\|I - 2P\| = 1$.

Equivalently : $\exists S = I - 2P : X^{**} \to X^{**}$, $\overbrace{}^{\text{isometric symmetry}}$ $/\!/ X$:

$$X^{**} = X \oplus_u X_s$$

$(\forall x \in X, \forall \sigma \in X_s) \qquad \|x + \sigma\| = \|x - \sigma\|$

<u>Remark</u>. P is unique.



<u>Proof</u>. $\exists \tilde{R}_n : X \longrightarrow X$, $rk(\tilde{R}_n) < \infty$:

(1) $\tilde{R}_n x \xrightarrow[n\to\infty]{} x$, $\forall x \in X$,

(2'') $\| I - 2\tilde{R}_n \| \leqslant 1 + \delta_n$, $\prod_{n \geqslant 1} (1+\delta_n) \leqslant 1+\varepsilon$,

(4) $\tilde{R}_m \tilde{R}_n = \tilde{R}_n$, $m > n$,

and $T_n = \tilde{R}_n - \tilde{R}_{n-1}$ $(\tilde{R}_0 = 0)$ satisfy :

(3) $\sup_{N \geqslant 1} \sup_{\varepsilon_n = \pm 1} \left\| \sum_{n=1}^{N} \varepsilon_n T_n \right\| \leqslant 1 + \varepsilon$.

For $x^{**} \in X^{**}$:

$$P x^{**} = \sum_{n=1}^{\infty} T_n^{**} x^{**} = \lim_{n\to\infty} \tilde{R}_n^{**} x^{**}$$

unconditionaly convergeant series <u>in X</u> since $X \not\supset c_0$.

- (4) $\Rightarrow$ P projection.
- $\| I - 2P \| \leqslant \liminf \| I - 2\tilde{R}_n^{**} \| \leqslant \liminf (1+\delta_n) = 1$

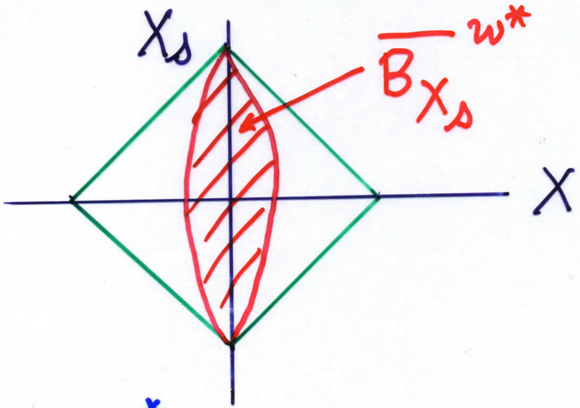
Theorem 2 X (UMAP), $X \not\supset c_0$

$$X^{**} = X \oplus_u X_s \Rightarrow X_s \ w^*\text{-closed}.$$

Banach–Dieudonné Theorem:

$$X_s \ w^*\text{-closed} \Leftrightarrow B_{X_s} \ w^*\text{-closed}$$

Lemma B_{X_s} w^*-closed $\Leftrightarrow X \cap \overline{B_{X_s}}^{w^*} = \{0\}$



Proof. $f \in \overline{B_{X_s}}^{w^*}$: $f = \underset{X}{\underset{\Uparrow}{x}} + \underset{X_s}{\underset{\Uparrow}{\sigma}}$

$\|I - 2P\| = 1 \Rightarrow \|P\|, \|I - P\| \leq 1 \Rightarrow \|\sigma\| \leq 1.$

Hence : $\sigma \in B_{X_s} \subset \overline{B_{X_s}}^{w^*}$

and : $\frac{x}{2} = \frac{1}{2}(f - \sigma) \in \overline{B_{X_s}}^{w^*} \cap X = \{0\} \Rightarrow$

$f = \sigma \in B_{X_s}$.

Proof of Th. 2. Let $x_0 \in X \cap \overline{B_{X_s}}^{w^*}$.

Local reflexivity : $(\forall \sigma \in B_{X_s})(\exists t_\alpha \in B_X)$

- $t_\alpha \xrightarrow{w^*} \sigma$
- $(\forall x \in X)$ $\lim_\alpha (\|x + t_\alpha\| - \|x - t_\alpha\|) = 0.$

Then :

$$\exists\, x_\beta \in B_X \quad : \quad x_0 = w\text{-}\lim x_\beta$$

and

$$(\forall x \in X) \qquad \lim_\beta \left(\|x + x_\beta\| - \|x - x_\beta\| \right) = 0 \qquad (*).$$

Lemma X (UMAP). For each bounded net $y_\beta \xrightarrow{w} 0$, we have :

$$(\forall y \in X) \qquad \lim_\beta \left(\|y + y_\beta\| - \|y - y_\beta\| \right) = 0 \qquad (**).$$

Apply the lemma with

$$y_\beta = x_0 - x_\beta \quad \text{and} \quad y = (2n-1)x_0 \quad :$$

$$\begin{aligned}
\overline{\lim}\, \|2n x_0 - x_\beta\| &= \overline{\lim}\, \|(2n-1)x_0 + (x_0 - x_\beta)\| \\
&\overset{(**)}{=} \overline{\lim}\, \|(2n-1)x_0 - (x_0 - x_\beta)\| \\
&= \overline{\lim}\, \|(2n-2)x_0 + x_\beta\| \\
&\overset{(*)}{=} \overline{\lim}\, \|(2n-2)x_0 - x_\beta\| ,
\end{aligned}$$

So :

$$\overline{\lim_\beta}\, \|2n x_0 - x_\beta\| = \overline{\lim_\beta}\, \|x_\beta\| , \quad \forall n \geq 1 ;$$

only possibility : $\underline{x_0 = 0}$.

Proof of the lemma. Let $R_n : X \to X$

(1) $R_n x \xrightarrow[n\to\infty]{} x , \quad \forall x \in X$

(2") $\|I - 2R_n\| \xrightarrow[n\to\infty]{} 0$.

Fix $\varepsilon > 0$, and $y \in X$.

$$(\exists n_0) \quad \begin{cases} \| y - R_{n_0} y \| \leq \frac{\varepsilon}{4} \\ \| I - 2R_{n_0} \| \leq 1 + \varepsilon \end{cases} .$$

$$\left.\begin{array}{l} y_\beta \xrightarrow{w} 0 \\ rk(R_{n_0}) < \infty \end{array}\right\} \Rightarrow \left[\begin{array}{ll} (\exists \beta_0) & \beta \geq \beta_0 \Rightarrow \\ & \| R_{n_0}(y_\beta) \| \leq \varepsilon/4 \end{array} \right].$$

Hence, for $\beta \geq \beta_0$:

$$\| y - R_{n_0}(y - y_\beta) \| \leq \varepsilon/2 ,$$

and

$$\| y + y_\beta + (I - 2R_{n_0})(y - y_\beta) \| = 2 \| y - R_{n_0}(y - y_\beta) \| \leq \varepsilon$$

so:

$$\overline{\lim} \left[\| y + y_\beta \| - \| y - y_\beta \| \right] \leq \varepsilon \left(1 + \overline{\lim_\beta} \| y - y_\beta \| \right).$$

Now

$$\left.\begin{array}{l} (y_\beta)_\beta \text{ bounded} \\ \varepsilon > 0 \text{ arbitrary} \\ y \longleftrightarrow (-y) \end{array}\right\} \rightarrow \overline{\lim} \left[\| y + y_\beta \| - \| y - y_\beta \| \right] = 0.$$

$\blacksquare$

Remark. We have $X = [(X_s)_\perp]^*$, and by [GKS], there is a commuting sequence $(U_n)_{n \geq 1}$ of finite rank operators $U_n : Y = (X_s)_\perp \to Y$ s.t. :

$$\begin{cases} (\forall y \in Y) & U_n y \xrightarrow[n \to \infty]{} y \\ (\forall x \in X) & U_n^* x \xrightarrow[n \to \infty]{} x \end{cases} .$$

Subspaces of L^1 with (UMAP).

$X \subset L^1 = L^1(0,1)$ with (UMAP)

$X \not\supset c_0 \longrightarrow X^{**} = X \oplus_u X_s$.

Hewitt-Yosida : $L^{1**} = L^1 \oplus_1 L^1_s$

Godefroy-Kalton-Saphar : $X^{\perp\perp} = X \oplus_1 (X^{\perp\perp} \cap L^1_s)$
(X is nicely placed in L^1)

Known facts :

- $E \subset L^1$
$E^{\perp\perp} = E \oplus_1 (E^{\perp\perp} \cap L^1_s) \iff B_E$ closed in measure
(Bukhvalov-Lozanovski)

Def : $E^{\#} = \{ \varphi \in E^* ;\ \varphi|B_E$ continuous in measure$\}$

- $E \subset L^1$, $E^{**} = E \oplus_1 E_s \Rightarrow E^{\#} = (E_s)_\perp \equiv (E_s)^\perp \cap E^*$

Consequences. $X \subset L^1$ with (UMAP)
X_s w^*-closed $\Rightarrow X \cong (X^{\#})^*$ and
the topology of convergence in measure on B_X
is finer than the $w^* = \sigma(X, X^{\#})$-topology.

Example. Reflexive subspaces of L^1 with
a 1-unconditional basis (i.e. ℓ_p, $1<p\leq 2$).
On a reflexive subspace of L^1:
convergence in measure = convergence in norm.
(Kadeč-Pełczynski); in particular: finer than w-topology

Opposite case : B_X compact in measure.

Theorem 3 $X \subset L^1$, with (AP).

The following assertions are equivalent:

1) X has (UMAP) and B_X relatively compact in measure,
2) B_X compact in measure and $X^\#$ separates X,
2') " " " " locally convex in measure,
2'') " " " " convergence in measure is weaker on B_X than $w = \sigma(X, X^*)$.
3) $(\forall \varepsilon > 0)(\exists X_\varepsilon$ w^*-closed subspace of $\ell_1)$ $\text{dist}(X, X_\varepsilon) \leqslant 1 + \varepsilon$.

Preceding remarks : 1) $\Rightarrow$ 2) $\Rightarrow$ 2') by compacity.
$\Rightarrow$ 2'')

2') $\Rightarrow$ 2) : quite easy.

2'') $\Rightarrow$ 2) : no proof given.

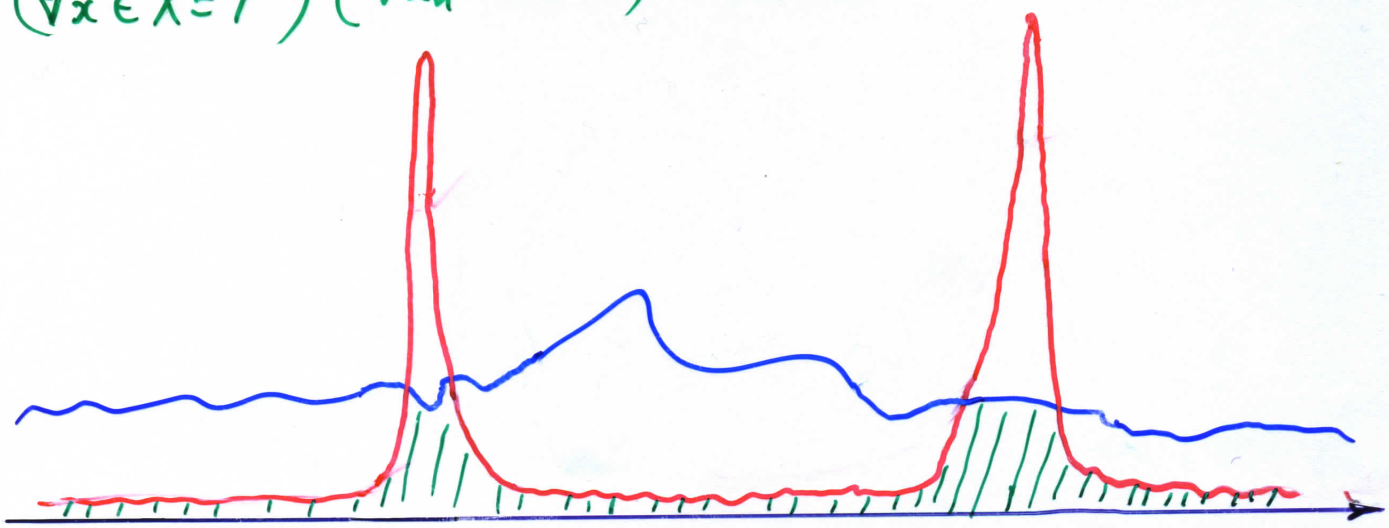
3) $\Rightarrow$ 1) : used Alspach : Z quotient of c_0 $\Rightarrow$ $(\forall \varepsilon > 0)(\exists Z_\varepsilon \subset c_0)$ $\text{dist}(Z, Z_\varepsilon) \leqslant 1 + \varepsilon$.

Proof of 2) $\Rightarrow$ 3) (sketched)

2) $\Rightarrow$ [$\sigma(X, X^\#)$ = convergence in measure on B_X]

$\Rightarrow$ $Y = X^\#$ has property (m_1^*) :

$(\forall x \in X = Y^*)(\forall x_n \xrightarrow{w^*} 0)$ $\overline{\lim} \|x + x_n\| = \|x\| + \overline{\lim} \|x_n\|$



Kalton - D. Werner (1993) Y separable Banach space, $Y \not\supset \ell_1$ and Y has (m_1^*); then $(\forall \varepsilon > 0)\ (\exists Y_\varepsilon \subset c_0)$ $\operatorname{dist}(Y, Y_\varepsilon) \leq 1+\varepsilon$.

We have to prove now:

Theorem 4 $Z \subset c_0$, and Z^* has (AP). Then $(\forall \varepsilon > 0)\ (\exists X_n \subset Z^*$ finite dimensional $)$

$$\exists T : Z^* \longrightarrow \Big(\bigoplus_{n \geq 1} X_n\Big)_{\ell_1} \quad w^*\text{-}w^*\text{- continuous}$$

$$(1-\varepsilon)\|x\| \leq \|Tx\| \leq (1+\varepsilon)\|x\|.$$

When $Z^* \subset L^1$; $X_n \overset{1+\varepsilon}{\hookrightarrow} \ell_1 \Rightarrow \Big(\bigoplus_{n \geq 1} X_n\Big)_{\ell_1} \overset{1+\varepsilon'}{\hookrightarrow} \ell_1$.

Proof of Th. 4 (idea).

Feder (convexity argument in $\mathcal{K}(Z, c_0)$):

$$\exists R_n : Z \longrightarrow Z \quad \operatorname{rk}(R_n) < \infty$$

(i) $z = \sum_{n=1}^{\infty} R_n z, \quad \forall z \in Z$

(ii) $z^* = \sum_{n=1}^{\infty} R_n^* z^*, \quad \forall z^* \in Z^*$

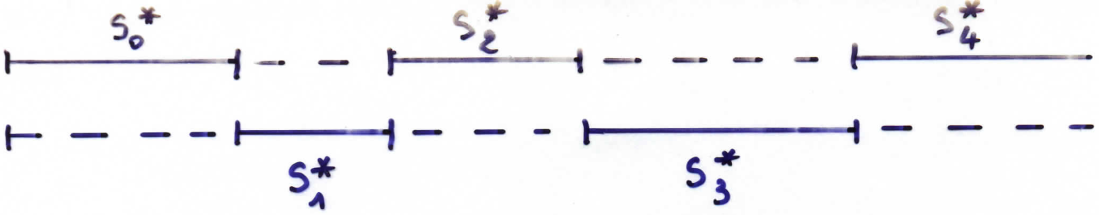
(iii) $\sup_{N \geq 1} \sup_{\varepsilon_n = \pm 1} \Big\| \sum_{n=1}^{N} \varepsilon_n R_n \Big\| \leq 1+\varepsilon$.

Skipped-blocking argument: $\exists$ integers: $0 = k_0 < k_1 < \dots$

$$S_j = \sum_{n=k_j+1}^{k_{j+1}} R_n$$

satisfy:

$$(\forall z^* \in Z^*) \begin{cases} \Big\| \sum_{j=0}^{J} S_{2j}^* z^* \Big\| \geq \frac{1}{1+\varepsilon} \sum_{j=0}^{J} \| S_{2j}^* z^* \| \\ \Big\| \sum_{j=0}^{J} S_{2j+1}^* z^* \Big\| \geq \frac{1}{1+\varepsilon} \sum_{j=0}^{J} \| S_{2j+1}^* z^* \| \end{cases}$$



By (iii), we have then:

$$\Big\| \sum_{j=1}^{J} s_j^* z^* \Big\| \geqslant \frac{1}{2(1+\varepsilon)} \Big(\Big\| \sum_{j \text{ even}} s_j^* z^* \Big\| + \Big\| \sum_{j \text{ odd}} s_j^* z^* \Big\| \Big)$$

$$\geqslant \frac{1}{2(1+\varepsilon)^2} \Big(\sum_{j \text{ even}} \| s_j^* z^* \| + \sum_{j \text{ odd}} \| s_j^* z^* \| \Big)$$

$$= \frac{1}{2(1+\varepsilon)^2} \sum_{j=1}^{J} \| s_j^* z^* \|.$$

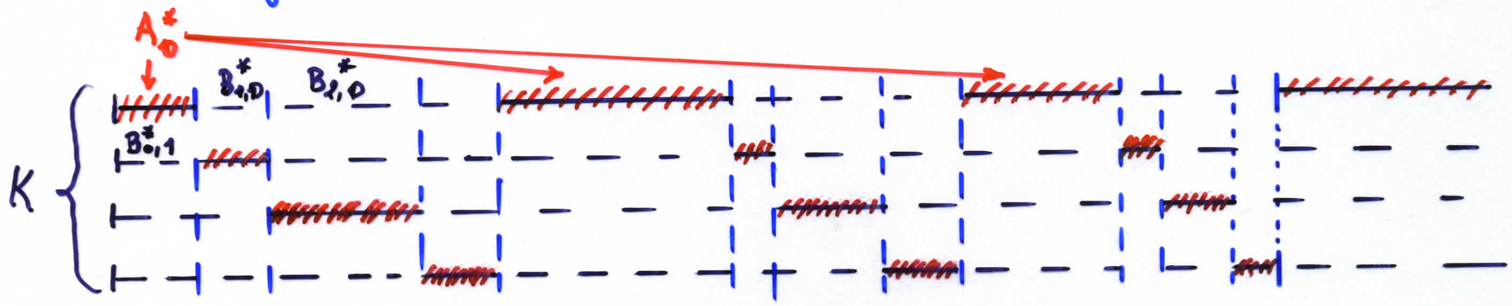
<u>Elimination of the constant 2</u>. s_j constructed s.t.

$\forall\, k_1 \leqslant k_2 < k_3 \leqslant k_4 < k_5 \leqslant \dots$

$$\Big\| \sum_{j=1}^{J} \Big(\sum_{k_{2j-1} \leqslant k \leqslant k_{2j}} s_k^* z^* \Big) \Big\| \geqslant \frac{1}{1+\varepsilon} \sum_{j=1}^{J} \Big\| \sum_{k_{2j-1} \leqslant k \leqslant k_{2j}} s_k^* z^* \Big\|$$

For $K = K(\varepsilon)$, we set $(0 \leqslant s \leqslant K-1)$:

$$A_s = \sum_{j \equiv s \,(\mathrm{mod}\, K)} s_j \qquad , \quad B_{k,s} = \sum_{(k-1)K + s < j < kK + s} s_j$$



Khintchine inequalities (when $Z^* \subset L^1$ for instance) give:

$$\sum_{s=0}^{K-1} \| A_s^* z^* \| \leqslant \sqrt{K} \Big(\sum_{s=0}^{K-1} \| A_s^* z^* \|^2 \Big)^{1/2} \leqslant \sqrt{2K}\,(1+\varepsilon)\, \| z^* \|$$

so:

$$\Big(1 - \sqrt{\frac{2}{K}}\,(1+\varepsilon) \Big) \| z^* \| \leqslant \frac{1}{K} \sum_{s=0}^{K-1} \sum_{k=0}^{+\infty} \| B_{k,s}^* z^* \| \leqslant (1+\varepsilon) \Big(1 + \sqrt{\frac{2}{K}}\,(1+\varepsilon) \Big) \| z^* \|.$$

A counterexample.

N. Kalton and D. Werner proved that for $X \subset L^p$, $1 < p < \infty$, $p \neq 2$, we have

B_X compact in measure $\iff$
$(\forall \varepsilon > 0)\ (\exists X_\varepsilon \subset \ell_p)$ $\operatorname{dist}(X, X_\varepsilon) \leqslant 1 + \varepsilon$.
So X has automatically (UMAP).
No more assumption is needed : on B_X

convergence in measure = convergence in $\|\cdot\|_1$

and so is locally convex.

This is not the case for $X \subset L^1$.

Theorem 5 $\exists E \subset L^1$, with (AP), B_E compact in measure, but E has not (UMAP).

Construction.

Bourgain-Rosenthal: $\exists$ integers $a_1 = 1 < a_2 < \cdots$
$\exists$ reals $p_k > 1$, $p_k \xrightarrow[k\to\infty]{} 1$
$\exists$ independant random variables Z_k s.t.

(i) $Z_i \overset{\text{dist.}}{=} |Y^{(k)}|$, $Y^{(k)}$ p_k-stable, $\|Y^{(k)}\|_1 = 1$, $\forall i \in I_k = [a_k, a_{k+1}[$

(ii) τ_m-dist $(V, \mathbb{R}.\mathbb{1}) \leq 1/2^k$, $\forall V = \sum\limits_{i\in I_k} \alpha_i (Z_i - \mathbb{1})$, $\|V\|_1 \leq 1$.

(iii) $\left\| \frac{1}{|I_k|} \sum\limits_{i\in I_k} Z_i - \mathbb{1} \right\|_1 \leq 1/k$.

We define:

$$E = \overline{\text{span}}\{\mathbb{1}, U_i, i \geq 1\}, \quad U_i = Z_i - \mathbb{1}$$

Independance of U_i, $i \geq 1$ and $E(U_i) = 0$ $\longrightarrow$
$\mathcal{B} = \{\mathbb{1}\} \cup \{U_i, i \geq 1\}$ unconditional basis of E.

(ii) $\longrightarrow$ B_E compact in measure

(i) $\longrightarrow$ $Z_i \xrightarrow[i\to\infty]{\text{meas.}} 0$

(iii) $\longrightarrow$ $\mathbb{1} \in (E^{\#})^{\perp}$ ▨

Remark $H^1(\mathbb{D}) \cong H^1 = \{ f \in L^1(\mathbb{T}) ; \hat{f}(n) = 0, \forall n < 0\}$ is a space such that:

- B_{H^1} is closed in measure and $(H^1)^{\#}$ separates H^1
- H^1 has an unconditional basis (Maurey, Carleson, Wojtaszczyk)

but H^1 has not (UMAP).