Composition operators: from dimension 1 to infinity (but not beyond)

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From works with

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First part: dimension 1

Hardy space H^2:

$$\mathbb{D} = \{z \in \mathbb{C}\,;\ |z| < 1\} \qquad f : \mathbb{D} \to \mathbb{C} \quad \text{analytic}$$

The space $H^2 = H^2(\mathbb{D})$ is that of the analytic functions f on $\mathbb{D}$ such that:

$$\|f\|_{H^2}^2 := \sup_{0<r<1} \int_0^{2\pi} |f(r\,\mathrm{e}^{it})|^2 \,\frac{dt}{2\pi} < +\infty$$

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$$\text{if } f(z) = \sum_{n=0}^{\infty} c_n z^n \text{ then } \|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |c_n|^2.$$

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Hence:

$$C_\varphi\colon H^2 \to H^2$$

is a bounded operator, called *composition operator* with symbol φ.

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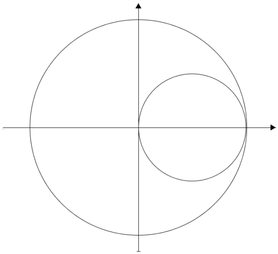
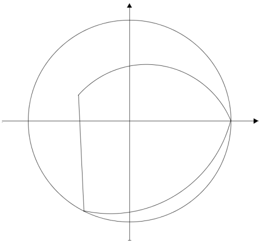
All these characterization says that C_φ is compact iff $\varphi(\mathbb{D})$ touches the unit circle (if it happens) only "sharply" (roughly speaking: non tangentially).
But actually, the knowledge of $\varphi(\mathbb{D})$ is not sufficient (unless φ is univalent).

Dimension 1

Compact composition operators



non compact



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They measure how much an operator is compact:

$$T \text{ compact} \iff a_n(T) \xrightarrow[n\to\infty]{} 0$$

$$T \in S_p(H) \iff \sum_n [a_n(T)]^p < \infty$$

GOAL

Give estimates on approximation numbers of composition operators

Lower estimates: no fast decay

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First stated by Parfenov (1988) under a "cryptic" form; explicitely by LQR (2012).

That use the fact that, if C_φ is compact, we can assume that $\varphi(0) = 0$ and $\varphi'(0) \neq 0$, and then the non-zero eigenvalues of C_φ are $[\varphi'(0)]^n$, $n = 0, 1, \ldots$, and:

Weyl Lemma

If $T : H \to H$ is a compact operator on a Hilbert space H and if $(\lambda_n)_{n\geq 1}$ is the sequence of the eigenvalues of T, numbered in non-increasing order, then:

$$\prod_{k=1}^{n} a_k(T) \geq \prod_{k=1}^{n} |\lambda_k|.$$

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It is actually the only case:

Theorem LQR 2012

If $a_n(C_\varphi) \lesssim r^n$ with $r < 1$, then $\|\varphi\|_\infty < 1$.

Two proofs:

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- The first proof uses Pietsch's factorization theorem

Pietsch's factorization theorem is used for getting a measure μ supported by a compact set $K \subseteq \varphi(\mathbb{D})$ such that, for the restriction operator $R_\mu \colon H \to L^2(\mu)$, one has $a_n(C_\varphi) \gtrsim a_n(R_\mu)$.

R_μ is more tractable than C_φ and it can be proved (assuming $\varphi(0) = 0$ for simplicity) that:

$$\|\varphi\|_\infty > r \quad \Longrightarrow \quad a_n(R_\mu) \gtrsim \frac{s^n}{\sqrt{n}} \quad \text{with } s = s(r) \xrightarrow[r \to 1]{} 1.$$

Two proofs:

- The first proof uses Pietsch's factorization theorem
- The second one uses the Green capacity of $\varphi(\mathbb{D})$

Theorem LQR 2014

If $\|\varphi\|_\infty < 1$, one has:

$$\lim_{n\to\infty} [a_n(C_\varphi)]^{1/n} = \mathrm{e}^{-1/\mathrm{Cap}\,[\varphi(\mathbb{D})]}$$

and (assuming $\varphi(0) = 0$ for simplicity):

$$\mathrm{Cap}\,[\varphi(\mathbb{D})] \xrightarrow[\|\varphi\|_\infty \to 1]{} +\infty.$$

Hence:

$$a_n(C_\varphi) \lesssim r^n \quad \text{implies} \quad \|\varphi\|_\infty < 1.$$

The proof is based on a theorem of H. Widom:

Theorem H. Widom (1972)

Let K be a compact set of $\mathbb{D}$ and

$$r_n = \sup_{f \in H^\infty,\, \|f\|_\infty \leq 1} \inf_R \|f - R\|_{\mathcal{C}(K)},$$

where the infimum is taken over all rational functions R with at most n poles, all outside of $\mathbb{D}$. Then:

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and of the fact (communicated by A. Ancona) that if V is an open connected set such that $\overline{V} \subset \mathbb{D}$, then $\mathrm{Cap}\,(V) = \mathrm{Cap}\,(\overline{V})$.

Lower estimates: slow decay

If $a_n(C_\varphi)$ cannot decay fast, it can decay arbitrarily slowly:

Theorem LQR 2012

For every vanishing non-increasing sequence $(\varepsilon_n)_{n\geq 1}$, there exists a symbol φ such that:

$$a_n(C_\varphi) \gtrsim \varepsilon_n \quad \text{for all } n \geq 1.$$

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That allows to have compact composition operators which are in no Schatten class S_p, $p < \infty$. This question of Sarason (1988) was first solved by Carroll and Cowen in 1991.

However, one has a better result:

Theorem H. Queffélec - K. Seip 2015

For every function $u\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $u(x) \searrow 0$ as $x \nearrow \infty$ and $u(x^2)/u(x)$ bounded below, there exists a compact composition operator such that:

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It is actually a corollary of a result which will be stated later.

More specific lower estimates

If $\mathbf{u} = (u_1, \ldots, u_n)$ is a finite sequence of complex numbers, its interpolation constant $M_{\mathbf{u}}$ is the smallest $M > 0$ such that:

$$\forall w_1, \ldots, w_n \text{ with } |w_j| \leq 1 \quad \exists f \in H^\infty \text{ with } \|f\|_\infty \leq M \text{ s.t.}$$

$$f(u_j) = w_j\,, j = 1, \ldots, n.$$

Proposition LQR 2013

Let φ a symbol, $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{D}^n$,
such that the $v_j = \varphi(u_j)$'s are distinct, and $\mathbf{v} = (v_1, \ldots, v_n)$.
Set:

$$\mu_n^2 = \inf_{1 \leq j \leq n} \frac{1 - |u_j|^2}{1 - |\varphi(u_j)|^2} \cdot$$

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where K_z is the reproducing kernel at z

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Then:

$$a_n(C_\varphi) \gtrsim \mu_n \, M_{\mathbf{v}}^{-2}.$$

That follows from the fact that if $\tilde{K}_{u_j} = K_{u_j}/\|K_{u_j}\|$, then:

$$M_{\mathbf{u}}^{-1}\Big(\sum_{j=1}^{n}|c_j|^2\Big)^{1/2} \leq \Big\|\sum_{j=1}^{n} c_j \tilde{K}_{u_j}\Big\|_{H^2} \leq M_{\mathbf{u}}\Big(\sum_{j=1}^{n}|c_j|^2\Big)^{1/2}.$$

By a suitable choice of $u_1, \dots, u_n$, we get then:

Theorem LQR 2013

Assume that $\varphi(]-1,1[) \subseteq \mathbb{R}$ and that $1 - \varphi(r) \leq \omega(1-r)$, $0 \leq r < 1$, where ω is continuous, increasing, sub-additive, vanishes at 0, and $\omega(h)/h \underset{h \to 0}{\longrightarrow} \infty$. Then:

$$a_n(C_\varphi) \gtrsim \sup_{0<s<1} \mathrm{e}^{-20/(1-s)} \sqrt{\frac{\omega^{-1}(as^n)}{as^n}}$$

where $a = 1 - \varphi(0) > 0$.

Examples

Lens maps

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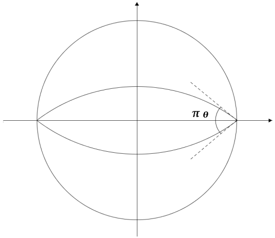

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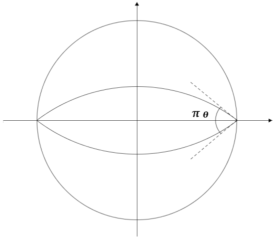

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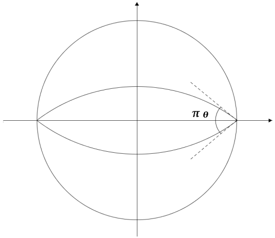

One has $\omega^{-1}(h) \approx h^{1/\theta}$; so we get:

$$a_n(C_{\lambda_\theta}) \geq \alpha\, \mathrm{e}^{-C\sqrt{n}}$$

where $\alpha, C > 0$ depends only on θ.

Corollary LLQR 2013

If φ is univalent and $\varphi(\mathbb{D})$ contains an angular sector centered on the unit cercle and with opening $\pi\theta$, $0 < \theta < 1$, then:

$$a_n(C_\varphi) \gtrsim \mathrm{e}^{-C\sqrt{n}}.$$

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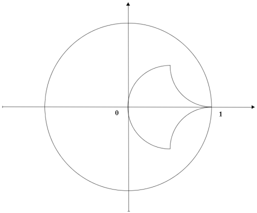

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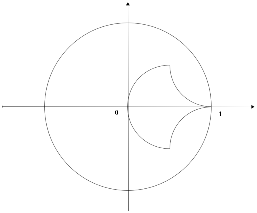

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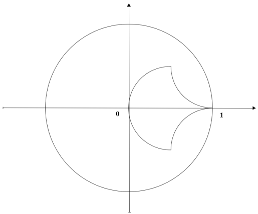

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$$f_\vartheta(z) = z(-\log z)^\vartheta \qquad \text{for } z \in V_\varepsilon$$

and
$$\varsigma_\vartheta = \exp(-f_\vartheta \circ g_\vartheta),$$

where g_ϑ is a conformal representation of $\mathbb{D}$ onto V_ε.

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Then $\omega(t) = t(\log 1/t)^\vartheta$ and $a_n(C_{\varsigma_\vartheta}) \gtrsim n^{-\vartheta/2}$ so that gives

$$C_{\varsigma_\vartheta} \in S_p \Rightarrow p > 2/\vartheta$$

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$$C_{\varsigma_\vartheta} \in S_p \Leftrightarrow p > 4/\theta.$$

Upper estimates

Proposition (Parfenov 1988)

For every symbol φ:

$$[a_n(C_\varphi)]^2 \lesssim \sup_{0<h<1,\, \xi\in\partial\mathbb{D}} \frac{1}{h} \int_{S(\xi,h)} |B(z)|^2 \, dm_\varphi(z)$$

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Follows from: 1) the subspace BH^2 has codimension $\leq n-1$, so $c_n(C_\varphi) \leq \| C_{\varphi|BH^2} \|$ where $c_n(C_\varphi)$ is the Gelfand number; and: 2) $a_n(C_\varphi) = c_n(C_\varphi)$ (because H^2 is a Hilbert space).

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We deduce from this proposition the following theorem.

Theorem LQR 2013

Assume that φ is continuous on $\overline{\mathbb{D}}$ and that $\varphi(\overline{\mathbb{D}})$ is contained in a polygon with vertices $\varphi(\mathrm{e}^{it_1}), \ldots, \varphi(\mathrm{e}^{it_N})$. Then, if:

$$|\varphi(\mathrm{e}^{it}) - \varphi(\mathrm{e}^{it_j})| \gtrsim \varpi(|t - t_j|),$$

for $|t - t_0|$ small enough and $j = 1, \ldots, N$, and ϖ is continuous, increasing, sub-additive, vanishes at 0, and $\varpi(h)/h \xrightarrow[h\to 0]{} \infty$, one has, for some constants $\kappa, \sigma > 0$:

$$a_n(C_\varphi) \lesssim \sqrt{\frac{\varpi^{-1}(\kappa\, 2^{-k_n})}{\kappa\, 2^{-k_n}}}\,,$$

where k_n is the largest integer such that $N\,k\,d_k < n$ and d_k is the integer part of $\sigma \log \frac{\kappa\, 2^{-n}}{\varpi(\kappa\, 2^{-n})} + 1$.

Examples

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Lens maps

For λ_θ, one has $N = 2$, $\varpi^{-1}(h) \approx h^{1/\theta}$, $d_k \approx k$ and $k_n \approx \sqrt{n}$; hence, for $\beta, c > 0$ depending only on θ:

$$a_n(C_{\lambda_\theta}) \leq \beta\, \mathrm{e}^{-c\sqrt{n}}.$$

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Remark. Similarly, H. Queffélec and K. Seip (2015) showed that if

$$\varphi(z) = \frac{1}{1 + (1 - z)^\alpha}, \qquad 0 < \alpha < 1,$$

then:

$$\mathrm{e}^{-\pi(1-\alpha)\sqrt{(2n)/\alpha}} \lesssim a_n(C_\varphi) \lesssim \mathrm{e}^{-\pi(1-\alpha)\sqrt{n/(2\alpha)}}$$

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For λ_θ, one has $N = 2$, $\varpi^{-1}(h) \approx h^{1/\theta}$, $d_k \approx k$ and $k_n \approx \sqrt{n}$; hence, for $\beta, c > 0$ depending only on θ:

$$a_n(C_{\lambda_\theta}) \leq \beta\, \mathrm{e}^{-c\sqrt{n}}.$$

Cusp map

For the cusp map χ, one has $N = 1$, $\varpi^{-1}(h) = \mathrm{e}^{-1/h}$, $d_k \approx 2^k$ and $2^{k_n} \approx n/\log n$; hence:

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The example of lens maps can be generalized as follows:

Theorem LQR 2013

If $\varphi(\mathbb{D})$ is contained in a polygon P with vertices on the unit circle, then, for constants $\alpha, \beta > 0$ depending only of P, one has:

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Spread lens maps

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(Theorem LLQR 2013)

Let λ_θ be a lens map and $\phi_\theta(z) = \lambda_\theta(z) \exp\big(-\frac{1+z}{1-z}\big)$. Then:

$$a_n(C_{\phi_\theta}) \lesssim (\log n/n)^{1/2\theta} \qquad n = 2, 3, \ldots$$

We get that $C_{\phi_\theta} \in S_p$ for $p > 2\theta$.

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Open question

Find a lower estimate for $a_n(C_{\phi_\theta})$.

Remark

Our proofs for the lower and upper estimates give in particular the following remark.

If B is a Blaschke product, $(BH^2)^{\perp}$ is the model space associated to B. One has:

Remark

For any symbol φ:

$$a_n(C_\varphi) \geq \sup_{u_1,\ldots,u_n\in(0,1)} \inf_{\substack{f\in(BH^2)^\perp \\ \|f\|=1}} \|C_\varphi^* f\|$$

where the supremum is taken over all Blaschke products with n zeros on the real interval $(0,1)$.

$$a_n(C_\varphi) \leq \inf_B \sup_{\substack{f\in BH^2 \\ \|f\|=1}} \|C_\varphi f\|$$

where the infimum is taken over all Blaschke products with less than n zeros.

Two sides estimates

To end this part, we state results of H. Queffélec and K. Seip with two sides estimates.

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Notation. Let $u\colon \overline{\mathbb{D}} \to \mathbb{R}$ be in the disk algebra (i.e. continuous on $\overline{\mathbb{D}}$ and analytic in $\mathbb{D}$) such that $u(\bar{z}) = u(z)$; let $\tilde{u}$ its harmonic conjugate, and:

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One assumes that, if $U(t) = u(\mathrm{e}^{it})$, U is increasing on$[0, \pi]$, $U(0) = 0$, and that U is smooth, except perhaps at $t = 0$. Moreover, one assumes that:

$$h_u(t) := \int_t^\pi \frac{U(x)}{x^2}\, dx \xrightarrow[t \to 0^+]{} \infty$$

to ensure that C_{φ_u} is compact.

Then:

Theorem (smooth case) H. Queffélec - K. Seip 2015

Assume that, for some $c > 1$ and $C > 0$, one has, for t small enough:

$$\frac{t\,U'(t)}{U(t)} \leq 1 + \frac{c}{|\log t|} \quad \text{and} \quad \frac{U(t)}{t\,h_u(t)} \leq \frac{C}{|\log t|(\log|\log t|)}$$

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These assumptions mean that φ_u is tangentially smooth at 1

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For the second result, we need a bit more notation. Writing:

$$U(t) = \mathrm{e}^{-\eta_U(|\log t|)} \qquad \text{for } 0 < t \leq 1 \text{ and } U(t) \leq 1/\mathrm{e},$$

one defines ω_U by the implicit equation:

$$\eta_U\big(x/\omega_U(x)\big) = \omega_U(x)$$

for $x \geq 0$ such that $\eta_U(x) \geq 1$.

Then:

Theorem (sharp cusp case) H. Queffélec - K. Seip 2015

Assume that:

$$\frac{\eta_U'(x)}{\eta_U(x)} = o\left(1/x\right) \quad \text{as } x \to \infty;$$

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The proofs are rather involved.

Second part: dimension $d \geq 2$

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Two domains are classical:

- the open ball
  $B_d = \{z = (z_1, \ldots, z_d) \in \mathbb{C}^d \,;\ |z_1|^2 + \cdots + |z_d|^2 < 1\}$

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In the sequel, we shall assume the symbol φ is such that $\varphi(\Omega)$ has non-void interior.

One has, as in dimension 1:

Proposition BLQR 2015

Let $C_\varphi \colon H^2(\Omega) \to H^2(\Omega)$ be compact ($\Omega = B_d$ or $\mathbb{D}^d$).
Let $\mathbf{u} = (u_1, \ldots, u_n) \in \Omega^n$ and $v_j = \varphi(u_j)$ be distinct. Let $M_{\mathbf{v}}$ be the interpolation constant of $\mathbf{v} = (v_1, \ldots, v_n)$. Then, setting:

$$\mu_n^2 = \inf_{1 \leq j \leq n} \prod_{k=1}^{d} \frac{1 - |u_{j,k}|^2}{1 - |v_{j,k}|^2} \, ,$$

one has:

$$a_n(C_\varphi) \gtrsim \mu_n M_{\mathbf{v}}^{-2} .$$

Then:

Theorem BLQR 2015

Let $C_\varphi \colon H^2(\Omega) \to H^2(\Omega)$ be compact ($\Omega = B_d$ or $\mathbb{D}^d$). Then, for some constant $C > 0$, one has:

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The interesting point is the dependence with the dimension d.

It is obtained with a good choice of the sequence $(u_1, \dots, u_n)$ in the previous proposition, and using estimates on its interpolation constant due to:

- P. Beurling when $\Omega = \mathbb{D}^d$;
- B. Berndtsson (1985) when $\Omega = B_d$.

Generalization

A bounded symmetric domain of $\mathbb{C}^d$ is a bounded open convex and circled subset Ω of $\mathbb{C}^d$ such that for every point $a \in \Omega$, there is an involutive bi-holomorphic map $u\colon \Omega \to \Omega$ such that a is an isolated fixed point of u (equivalently, as shown by J.-P. Vigué: $u(a) = a$ and $u'(a) = -id$).

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The unit ball B_d and the polydisk $\mathbb{D}^d$ are examples of bounded symmetric domains.

Hardy space

The Shilov boundary S_Ω of Ω is the smallest closed set $F \subseteq \partial\Omega$ such that

$$\sup_{z \in \overline{\Omega}} |f(z)| = \sup_{z \in \Omega} |f(z)|$$

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But the Shilov boundary of the bidisk $\mathbb{D}^2$ is

$$\{(z_1, z_2) \in \mathbb{C}^2 \quad |z_1| = |z_2| = 1\},$$

though
$\partial\mathbb{D}^2 = \{(z_1, z_2) \in \mathbb{C}^2 \, ; \; |z_1|, |z_2| \leq 1 \text{ and } |z_1| = 1 \text{ or } |z_2| = 2\}$.

There is a unique probability measure σ on S_Ω which is invariant by the automorphisms u of Ω such that $u(0) = 0$.

The Hardy space $H^2(\Omega)$ is the space of analytic functions $f : \Omega \to \mathbb{C}$ such that:

$$\|f\|_2 = \left(\sup_{0<r<1} \int_{S_\Omega} |f(r\xi)|^2 \, d\sigma(\xi) \right)^{1/2} < \infty \, .$$

We have:

Theorem BLQR 2015

Let Ω be a bounded symmetric domain of $\mathbb{C}^d$ and $C_\varphi\colon H^2(\Omega) \to H^2(\Omega)$ compact. Then, for some constant $C > 0$, one has:

$$a_n(C_\varphi) \gtrsim \mathrm{e}^{-Cn^{1/d}}.$$

That use Weyl Lemma and:

Theorem (D. Clahane 2005)

Let Ω be a bounded symmetric domain of $\mathbb{C}^d$ and $\varphi \colon \Omega \to \Omega$ be a holomorphic map inducing a compact composition operator $C_\varphi \colon H^2(\Omega) \to H^2(\Omega)$. Then φ has a unique fixed point $z_0 \in \Omega$ and the spectrum of C_φ consists of 0, and all possible products of eigenvalues of the derivative $\varphi'(z_0)$.

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However, the first proof give more information to construct examples.

Upper estimates

Theorem BLQR 2015

Let $\Omega = B_{d_1} \times \cdots \times B_{d_N}$, $d_1 + \cdots + d_N = d$.

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$N = 1$ gives the ball B_d

$N = d$ and $d_1 = \cdots = d_N = 1$ give the polydisk $\mathbb{D}^d$.

Upper estimates

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Let $\Omega = B_{d_1} \times \cdots \times B_{d_N}$, $d_1 + \cdots + d_N = d$.

If $\|\varphi\|_\infty < 1$, then C_φ is compact and $a_n(C_\varphi) \lesssim \mathrm{e}^{-Cn^{1/d}}$.

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Does that hold for Ω a general bounded symmetric domain?

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Open question

Does the converse hold?

In the case of the polydisk $\Omega = \mathbb{D}^d$, and "diagonal" symbols, one has:

Theorem BLQR 2015

Let $\varphi_1, \ldots, \varphi_D \colon \mathbb{D} \to \mathbb{D}$ be symbols inducing compact composition operators on $H^2(\mathbb{D})$, and let:

$$\varphi(z_1, \ldots, z_d) = \big(\varphi_1(z_1), \ldots, \varphi_d(z_d)\big).$$

Then, for $C_\varphi \colon H^2(\mathbb{D}^d) \to H^2(\mathbb{D}^d)$, one has:

$$a_n(C_\varphi) \leq \left(2^{d-1} \prod_{j=1}^{d} \|C_{\varphi_j}\|\right) \inf_{n_1 \cdots n_d \leq n} \left[a_{n_1}(C_{\varphi_1}) + \cdots + a_{n_d}(C_{\varphi_d})\right].$$

To prove that, for fixed $n_1, \ldots, n_d$ such that $n_1 \cdots n_d \leq n$, one consider, for each $j = 1, \ldots, d$, an operator

$$R_j \colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$$

with rank $< n_j$ such that $\| C_{\varphi_j} - R_j \| = a_{n_j}(C_{\varphi_j})$.

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One defines $R \colon H^2(\mathbb{D}^d) \to H^2(\mathbb{D}^d)$ by:

$$R(z^\alpha) = R_1(z_1^{\alpha_1}) \cdots R_d(z_d^{\alpha_d}) \,,$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$.

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where $\alpha = (\alpha_1, \ldots, \alpha_d)$.

Then R has rank $< n_1 \cdots n_d \leq n$ and $\| C_\varphi - R \|$ is less than the upper estimate given in the theorem.

Examples

Multi-lens maps

Let $0 < \theta_1, \ldots, \theta_d < 1$ and $\lambda_{\theta_1}, \ldots, \lambda_{\theta_d}$ be the associated lens maps. Then, if:

$$\varphi(z_1, \ldots, z_d) = \big(\lambda_{\theta_1}(z_1), \ldots, \lambda_{\theta_d}(z_d)\big),$$

one has:

$$\mathrm{e}^{-\alpha n^{1/(2d)}} \lesssim a_n(C_\varphi) \lesssim \mathrm{e}^{-\beta n^{1/(2d)}}.$$

Multi-cusp map

Let χ be the above cusp map, and $\varphi(z_1, \ldots, z_d) = \big(\chi(z_1), \ldots, \chi(z_d)\big)$. Then:

$$e^{-\alpha n^{1/d} / \log n} \lesssim a_n(C_\varphi) \lesssim e^{-\beta n^{1/d} / \log n}.$$

Another type of example

Let $c_1, \ldots, c_d > 0$ such that $c_1 + \cdots + c_d \leq 1$ and $\varphi(z_1, \ldots, z_d) = (c_1 z_1 + \cdots + c_d z_d, 0, \ldots, 0)$. Then:

$$a_n(C_\varphi) \approx \frac{(c_1 + \cdots + c_d)^n}{n^{(d-1)/4}} \,.$$

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$$a_n(C_\varphi) \approx \frac{(c_1 + \cdots + c_d)^n}{n^{(d-1)/4}} \cdot$$

In particular, if $c_1 + \cdots + c_d = 1$, then C_φ is compact, and

$$C_\varphi \in S_p \quad \Leftrightarrow \quad p > 4/(d-1).$$

This example is called by Hervé "toy example"

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Third part: infinite dimension

Introduction

We saw that:

Theorem

For $C_\varphi \colon H^2(\mathbb{D}^d) \to H^2(\mathbb{D}^d)$, one has:

- always $a_n(C_\varphi) \gtrsim \mathrm{e}^{-C\, n^{1/d}}$
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As $\mathrm{e}^{-C\, n^{1/d}} \underset{d \to \infty}{\longrightarrow} \mathrm{e}^{-C} > 0$, one might believe that there is no compact composition operator in infinite dimension.

Actually, it is not the case, as we shall see.

Hardy space

We consider the infinite polydisk $\mathbb{D}^\infty$.
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It is natural to ask that it is the space of all functions f with

$$(1) \qquad f(z) = \sum_{\alpha \geq 0} c_\alpha z^\alpha \quad \text{and} \quad \|f\|_2^2 := \sum_{\alpha \geq 0} |c_\alpha|^2 < \infty \, ,$$

where $\alpha = (\alpha_j)_{j \geq 1}$, $z = (z_j)_{j \geq 1}$ and $z^\alpha = \prod_{j \geq 1} z_j^{\alpha_j}$.
If one asks absolute convergence in (1), we should have $\sum_{\alpha \geq 0} |z^\alpha|^2 < \infty$.

Since one has the Euler type formula:

$$\sum_{\alpha \geq 0} |z^\alpha|^2 = \prod_{j=1}^{\infty} \frac{1}{1 - |z_j|^2}$$

we get that:

$$\sum_{j=1}^{\infty} |z_j|^2 < \infty \, .$$

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we get that:

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Hence it is natural to consider $\Omega_2 = \mathbb{D}^\infty \cap \ell_2$ instead of the whole polydisk.

Actually, we will work with $\Omega_1 = \mathbb{D}^\infty \cap \ell_1$ (which is an open subset of ℓ_1) because of the following proposition:

Proposition LQR 2016

Let $\varphi_j \colon \mathbb{D} \to \mathbb{D}$ be analytic, $j = 1, 2, \ldots$ and $\varphi(z) = \big(\varphi_j(z_j)\big)_{j\geq 1}$. Then $\|C_\varphi(f)\|_2 < \infty$ for all $\|f\|_2 < \infty$ if and only if:

$$\sum_{j=1}^{\infty} |\varphi_j(0)| < \infty \, .$$

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Hence $H^2 = H^2(\Omega_1)$ will be the space of all $f \colon \Omega_1 \to \mathbb{C}$ such that $\|f\|_2 < \infty$.

Composition operators

We will say that φ is truly infinite-dimensional if $\varphi'(a)\colon \ell_1 \to \ell_1$ is one-to-one for some $a \in \Omega_1$.

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We will say that φ is truly infinite-dimensional if $\varphi'(a)\colon \ell_1 \to \ell_1$ is one-to-one for some $a \in \Omega_1$.

First, if $\varphi(\Omega_1)$ remains far from $\partial\Omega_1$, one has:

Theorem LQR 2016

Let $\varphi\colon \Omega_1 \to \Omega_1$ truly infinite-dimensional such that $\overline{\varphi(\Omega_1)} \subset \Omega_1$ is compact. Then:

1) $C_\varphi\colon H^2(\Omega_1) \to H^2(\Omega_1)$ is bounded, and even compact;
2) $\varphi'(0)\colon \ell_1 \to \ell_1$ is compact;
3) for all $p > 0$, one has:

$$\sum_{n=1}^{\infty} \frac{1}{\left[\log\left(1/a_n(C_\varphi)\right)\right]^p} = \infty.$$

Caution

There exist compact composition operators $C_\varphi \colon H^2(\Omega_1) \to H^2(\Omega_1)$ such that $\varphi(\Omega_1)$ is unbounded in ℓ_1.

One can take a diagonal symbol.

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Remark. Assuming $\overline{\varphi(\Omega_1)}$ compact in ℓ_1 instead compact in Ω_1 is not sufficient.

Example: $\varphi(z) = \Big(\dfrac{1+z_1}{2}, 0, 0, \ldots\Big)$.

Proposition LQR 2016

Let $|\lambda_1|, |\lambda_2|, \ldots < 1$ and $\varphi(z) = (\lambda_j z_j)_{j\geq 1}$. Then $\varphi\colon \Omega_1 \to \Omega_1$ and, for every $p > 0$:

$$(\lambda_j)_{j\geq 1} \in \ell_p \quad \Rightarrow \quad C_\varphi \in S_p.$$

In particular, there exist truly infinite-dimensional symbols on Ω_1 such that C_φ is in all Schatten classes S_p, $p > 0$.

Theorem LQR 2016

For every $0 < \delta < 1$, there exist compact composition operators on $H^2(\Omega_1)$ with truly infinite-dimensional symbol such that:

$$a_n(C_\varphi) \lesssim \exp\left[- c\, \mathrm{e}^{b(\log n)^\delta} \right].$$

That's all Folks !

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