NON LINEAR GEOMETRY OF BANACH SPACES

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The fundamental problem in non linear Banach space theory is to describe how the linear structure of a Banach space is (or is not) determined by its linear structure. In other words, we try to exhibit the linear properties of Banach spaces that are stable under some particular non linear maps. These non linear maps can be of very different nature: Lipschitz isomorphisms or embeddings, uniform homeomorphisms, uniform or coarse embeddings.

It is often said that the birth of the subject coincides with the famous theorem by Mazur and Ulam [5] in 1932 who showed that any onto isometry between two normed spaces is affine. Much later, another very important event for this area was the publication in 2000 of the authoritative book by Benyamini and Lindenstrauss [2]. Since then, there has been a lot of progress in various directions of this field.

In this series of lectures we will concentrate on uniform homeomorphisms and more generally on coarse-Lipschitz embeddings between Banach spaces (roughly speaking, a coarse-Lipschitz embedding is a map which is bi-Lipschitz for large distances). We will first spend some time giving the basic definitions and properties. We will then survey a few classical results. In particular, we will explain why the local properties of Banach spaces are preserved under such operations (by local properties of Banach spaces, we mean properties of their finite dimensional subspaces). We will also study Aharoni's theorem [1] which asserts that every separable metric space is Lipschitz embeddable into c_0. We will see a few recent variants of it and address the question of its converse: what can be said about the linear properties of a Banach space in which c_0 embeds non linearly? Finally we will focus on the links between the metric and the linear asymptotic structure of Banach spaces. It follows from the work of N.J. Kalton and others that the existence of an asymptotically uniformly smooth norm on a Banach spaces is preserved under some non linear transformations. This has been known for a few years and we will explain it. Very recently N.J. Kalton [4] made a very important progress in the other direction by showing that some kind of asymptotic uniform convexity is stable under coarse-Lipschitz embeddings. He was able to derive striking applications. Our aim is to give an account of these results.

As it is clear from this abstract, a great part of this course will be based on Nigel Kalton's work. Unfortunaltely, Nigel Kalton passed away in August. This is a terrible loss for his family, his friends and for mathematics, but we will try to enjoy the wonderful ideas that he has left for the researchers in this field.

References

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