Theory of Hörmander multiplier theorems. Dilations of polynomially bounded groups

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Let $T(n)$ be a family of linear operators acting on some Banach space X. Let Y be a further space, and $J : X \to Y$ (resp $P : Y \to X$) be an injection (resp. a surjection) between the two. One says that $T(n)$ possesses a dilation to $W(n)$, an operator family acting on Y, if the following diagram commutes for any n :

$$\begin{array}{ccc} X & \xrightarrow{T(n)} & X \\ {\scriptstyle J}\downarrow & & \uparrow{\scriptstyle P} \\ Y & \xrightarrow[W(n)]{} & Y \end{array}$$

Dilations are used to transfer nice properties from $W(n)$ of a simpler structure, to general $T(n)$.

A classical example is the theorem of Stinespring, which states that $T(a) = V^*\pi(a)V$, where $T\colon A \to B(H)$ is a given completely bounded operator, and $\pi : A \to B(K)$ is a *-representation of A over a second Hilbert space $K = V(H)$.

A second example is the dilation by similarity $S \in B(H)^{-1}$, of bounded strongly continuous semigroups $(T(t))_{t\geq 0}$, to a contractive semigroup $(W(t))_{t\geq 0}$ (result of Le Merdy), valid if the generator has bounded imaginary powers, or equivalently, a bounded H^∞ calculus.

We present a dilation result for polynomially bounded groups $T(t)$ having a bounded H^∞ calculus. The underlying space X has finite cotype, i.e. does not contain the l_n^∞'s uniformly for all dimensions n. Such groups occur naturally as $T(t) = A^{it}$, where A is a differential operator of Laplacian type.

For example,

$$\|(-\Delta)^{it}\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \leq C_p(1+|t|)^{d|\frac{1}{2}-\frac{1}{p}|} \quad (1<p<\infty)$$

by a result of Sikora and Wright.

The dilated $W(t)$ will have the same polynomial bound and moreover be a shift group. Thus, resolvents and other operators affiliated to the group are also of particularly simple nature, making estimates easier.

We then apply our result to the functional calculus of such a group. This is where type and cotype of the space X come into play. We show how our calculus compares to Hörmander's classical spectral multiplier theorem in the case $T(t) = (-\Delta)^{it}$.