Inversion of the divergence on bounded domains of $\mathbb{R}^n$

Emmanuel Russ

Abstract

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of $\mathbb{R}^n$ ($n \geq 1$) and $p \in [1, +\infty]$. Denote by $\text{div}$ the divergence operator, namely, if $u = (u_1, \ldots, u_n) : \Omega \to \mathbb{R}^n$ is a vector field,

$$\text{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i}.$$ 

Let $f \in L^p(\Omega)$. We are interested in the following two questions:

1. Is it possible to find a vector field $u$ on $\Omega$ belonging to the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n)$ such that

$$\text{div} u = f \ \text{in} \ \Omega ? \quad (0.1)$$

2. Assume furthermore that $\int_\Omega f(x) dx = 0$. Is it possible to find a vector field $u$ on $\Omega$ belonging to the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n)$, solving (0.1) and satisfying furthermore

$$u \cdot \nu = 0 \ \text{on} \ \partial \Omega \quad (0.2)$$

or

$$u = 0 \ \text{on} \ \partial \Omega ? \quad (0.3)$$

In this course, we investigate these two questions. As we shall see, the answer to 1. is positive when $1 < p < +\infty$ and negative if $p = 1$ or $p = +\infty$. Question 2. is much more delicate (in particular, one has to give a precise meaning to conditions (0.2) and (0.3)), and the answer involves some geometric properties of $\Omega$. We will present various results and several approaches for question 2. We rely on basic properties of Sobolev spaces, which can be found, for instance, in [1] (Chapitres 8 et 9) or [2] (Chapter 5).

References
