Inversion of the divergence on bounded domains of $\mathbb{R}^n$

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Abstract

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ and $p \in [1, +\infty]$. Denote by div the divergence operator, namely, if $\mathbf{u} = (u_1, ..., u_n) : \Omega \to \mathbb{R}^n$ is a vector field,

$$\text{div}\mathbf{u} = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i}.$$

Let $f \in L^p(\Omega)$. We are interested in the following two questions:

1. Is it possible to find a vector field $\mathbf{u}$ on Ω belonging to the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n)$ such that
$$\text{div}\mathbf{u} = f \text{ in } \Omega \ ? \tag{0.1}$$
2. Assume furthermore that $\int_\Omega f(x)dx = 0$. Is it possible to find a vector field $\mathbf{u}$ on Ω belonging to the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n)$, solving (0.1) and satisfying furthermore
$$\mathbf{u} \cdot \nu = 0 \text{ on } \partial\Omega \tag{0.2}$$
or
$$\mathbf{u} = 0 \text{ on } \partial\Omega \ ? \tag{0.3}$$

In this course, we investigate these two questions. As we shall see, the answer to 1. is positive when $1 < p < +\infty$ and negative if $p = 1$ or $p = +\infty$. Question 2. is much more delicate (in particular, one has to give a precise meaning to conditions (0.2) and (0.3)), and the answer involves some geometric properties of Ω. We will present various results and several approaches for question 2. We rely on basic properties of Sobolev spaces, which can be found, for instance, in [1] (Chapitres 8 et 9) or [2] (Chapter 5).

References

[1] H. Brezis, *Analyse fonctionnelle: Théorie et applications*, Dunod, 2005.

[2] L. C. Evans, *Partial Differential Equations*, Amer. Math. Soc., 1998.