JOURNAL OF
FUNCTION SPACES AND APPLICATIONS
Volume 6, Number 3 (2008), 277-292


http://www.jfsa.net

# A criterion of weak compactness for operators on subspaces of Orlicz spaces

**Pascal Lefèvre, Daniel Li,**
**Hervé Queffélec and Luis Rodríguez-Piazza**

(*Communicated by Nigel Kalton*)

**2000 Mathematics Subject Classification.** Primary: 46E30; Secondary: 46B20.

**Keywords and phrases.** Morse-Transue space, Orlicz space, weakly compact operators.

**Abstract.** We give a criterion of weak compactness for the operators on the Morse-Transue space $M^\Psi$, the subspace of the Orlicz space $L^\Psi$ generated by $L^\infty$.

## 1. Introduction and Notation

In 1975, C. Niculescu established a characterization of weakly compact operators $T$ from $\mathcal{C}(S)$, where $S$ is a compact space, into a Banach space $Z$ ([14, 15], see [3] Theorem 15.2 too): $T\colon \mathcal{C}(S) \to Z$ is weakly compact if and only if there exists a Borel probability measure $\mu$ on $S$ such that, for every $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$ such that:

$$\|Tf\| \leq C(\epsilon)\, \|f\|_{L^1(\mu)} + \epsilon\, \|f\|_\infty\,, \quad \forall f \in \mathcal{C}(S).$$

The same kind of result was proved by H. Jarchow for $\mathbb{C}^*$-algebras in [7], and by the first author for $A(\mathbb{D})$ and $H^\infty$ (see [11]). The criterion for $H^\infty$

played a key role to give an elementary proof of the equivalence between weak compactness and compactness for composition operators on $H^\infty$.

Beside these spaces, one natural class of Banach spaces is the class of Orlicz spaces $L^\Psi$. Unfortunately, we shall see that the above criterion is in general not true for Orlicz spaces. However, it remains true when we restrict ourselves to subspaces of the Morse-Transue space $M^\Psi$. This space is the closure of $L^\infty$ in the Orlicz space $L^\Psi$.

In this paper, we first give a characterization of the operators from a subspace of $M^\Psi$ which fix no copy of $c_0$. When the complementary function of $\Psi$ satisfies $\Delta_2$, that gives a criterion of weak compactness. If moreover $\Psi$ satisfies a growth condition, that we call $\Delta^0$, the criterion has a more usable formulation, analogous to those described above.

As in the case of $H^\infty$ (but this is far less elementary), this new version obtained for subspaces of Morse-Transue spaces (Theorem 4), combined with a study of generalized Carleson measures, may be used to prove the equivalence between weak compactness and compactness for composition operators on Hardy-Orlicz spaces (see [13]), when $\Psi$ satisfies $\Delta^0$.

However, we think that this characterization has an intrinsic interest for Orlicz spaces, and will be useful not only for composition operators (see Remark 5 at the end of the paper).

In this note, we shall consider Orlicz spaces defined on a probability space $(\Omega, \mathbb{P})$, that we shall assume non purely atomic.

By an Orlicz function, we shall understand that $\Psi\colon [0,\infty] \to [0,\infty]$ is a non-decreasing convex function such that $\Psi(0) = 0$ and $\Psi(\infty) = \infty$. To avoid pathologies, we shall assume that we work with an Orlicz function $\Psi$ having the following additional properties: $\Psi$ is continuous at $0$, strictly convex (hence *strictly* increasing), and such that

$$\frac{\Psi(x)}{x} \underset{x\to\infty}{\longrightarrow} \infty.$$

This is essentially to exclude the case of $\Psi(x) = ax$. The Orlicz space $L^\Psi(\Omega)$ is the space of all (equivalence classes of) measurable functions $f\colon \Omega \to \mathbb{C}$ for which there is a constant $C > 0$ such that

$$\int_\Omega \Psi\Big(\frac{|f(t)|}{C}\Big)\, d\mathbb{P}(t) < +\infty$$

and then $\|f\|_\Psi$ (the *Luxemburg norm*) is the infimum of all possible constants $C$ such that this integral is $\leq 1$.

To every Orlicz function is associated the complementary Orlicz function $\Phi = \Psi^*\colon [0,\infty] \to [0,\infty]$ defined by:

$$\Phi(x) = \sup_{y\geq 0} \big(xy - \Psi(y)\big).$$

The extra assumptions on $\Psi$ ensure that $\Phi$ is itself strictly convex.

Throughout this paper, we shall assume, except explicit mention of the contrary, that the *complementary* Orlicz function satisfies the $\Delta_2$ condition ($\Phi \in \Delta_2$), *i.e.*, for some constant $K > 0$, and some $x_0 > 0$, we have:

$$\Phi(2x) \leq K\,\Phi(x), \qquad \forall x \geq x_0.$$

This is usually expressed by saying that $\Psi$ satisfies the $\nabla_2$ condition ($\Psi \in \nabla_2$). This is equivalent to say that for some $\beta > 1$ and $x_0 > 0$, one has $\Psi(x) \leq \Psi(\beta x)/(2\beta)$ for $x \geq x_0$, and that implies that $\frac{\Psi(x)}{x} \underset{x\to\infty}{\longrightarrow} \infty$. In particular, this excludes the case $L^\Psi = L^1$.

When $\Phi$ satisfies the $\Delta_2$ condition, $L^\Psi$ is the dual space of $L^\Phi$.

We shall denote by $M^\Psi$ the closure of $L^\infty$ in $L^\Psi$. Equivalently (see [16], page 75), $M^\Psi$ is the space of (classes of) functions such that:

$$\int_\Omega \Psi\Big(\frac{|f(t)|}{C}\Big)\,d\mathbb{P}(t) < +\infty, \qquad \forall C > 0.$$

This space is the *Morse-Transue space* associated to $\Psi$, and $(M^\Psi)^* = L^\Phi$, isometrically if $L^\Phi$ is provided with the Orlicz norm, and isomorphically if it is equipped with the Luxemburg norm (see [16], Chapter IV, Theorem 1.7, page 110).

We have $M^\Psi = L^\Psi$ if and only if $\Psi$ satisfies the $\Delta_2$ condition, and $L^\Psi$ is reflexive if and only if both $\Psi$ and $\Phi$ satisfy the $\Delta_2$ condition. When the complementary function $\Phi = \Psi^*$ of $\Psi$ satisfies it (but $\Psi$ does not satisfy this $\Delta_2$ condition, to exclude the reflexive case), we have (see [16], Chapter IV, Proposition 2.8, page 122, and Theorem 2.11, page 123):

$$(*) \qquad (L^\Psi)^* = (M^\Psi)^* \oplus_1 (M^\Psi)^\perp,$$

or, equivalently, $(L^\Psi)^* = L^\Phi \oplus_1 (M^\Psi)^\perp$, isometrically, with the Orlicz norm on $L^\Phi$.

For all the matter about Orlicz functions and Orlicz spaces, we refer to [16], or to [9].

## 2. Main result

Our goal in this section is the following criterion of weak compactness for operators. We begin with:

**Theorem 1.** *Let $\Psi$ be an arbitrary Orlicz function, and let $X$ be a subspace of the Morse-Transue space $M^\Psi$. Then an operator $T\colon X \to Y$ from $X$ into a Banach space $Y$ fixes no copy of $c_0$ if and only if for each $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\|Tf\| \leq \left[C_\varepsilon \int_\Omega \Psi\Big(\varepsilon \frac{|f|}{\|f\|_\Psi}\Big)\, d\mathbb{P} + \varepsilon\right] \|f\|_\Psi, \quad \forall f \in X. \tag{1}$$

Recall that saying that $T$ fixes a copy of $c_0$ means that there exists a subspace $X_0$ of $X$ isomorphic to $c_0$ such that $T$ realizes an isomorphism between $X_0$ and $T(X_0)$.

Before proving that, we shall give some consequences. First, we have

**Corollary 2.** *Assume that the complementary function of $\Psi$ has $\Delta_2$ ($\Psi \in \nabla_2$). Then for every subspace $X$ of $M^\Psi$, and every operator $T\colon X \to Y$, $T$ is weakly compact if and only if it satisfies* (1).

*Proof.* When the complementary function of $\Psi$ has $\Delta_2$, one has the decomposition ($*$), which means that $M^\Psi$ is $M$-ideal in its bidual (see [6, Chapter III]); this result was first shown by D. Werner [17] (see also [6, Chapter III, Example 1.4 (d), page 105]) by a different way, using the ball intersection property; note that in these references, it is moreover assumed that $\Psi$ does not satisfy the $\Delta_2$ condition, but if it satisfies it, the space $L^\Psi$ is reflexive, and so the result is obvious. But every subspace $X$ of a Banach space which is $M$-ideal of its bidual has Pełczyńki's property $(V)$ ([4, 5]; see also [6], Chapter III, Theorem 3.4), which means that operators from $X$ are weakly compact if and only if they fix no copy of $c_0$. □

With $\Psi$ satisfying the following growth condition, the characterization (1) takes on a more usable form.

**Definition 3.** *We say that the Orlicz function $\Psi$ satisfies the $\Delta^0$* condition *if for some $\beta > 1$*

$$\lim_{x\to+\infty} \frac{\Psi(\beta x)}{\Psi(x)} = +\infty.$$

This growth condition is a strong negation of the $\Delta_2$ condition and it implies that the complementary function $\Phi = \Psi^*$ of $\Psi$ satisfies the $\Delta_2$ condition.

Note that in the following theorem, we cannot content ourselves with $\Psi \notin \Delta_2$ (*i.e.* $\limsup_{x\to+\infty} \Psi(\beta x)/\Psi(x) = +\infty$), instead of $\Psi \in \Delta^0$ (see

Remark 3 in Section 3). An interesting question is whether the condition $\Psi \in \Delta^0$ is actually necessary for this characteriztion.

**Theorem 4.** *Assume that $\Psi$ satisfies the $\Delta^0$ condition, and let $X$ be a subspace of $M^\Psi$. Then every linear operator $T$ mapping $X$ into some Banach space $Y$ is weakly compact if and only if for some (and then for all) $1 \leq p < \infty$ and all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\text{(W)} \qquad \|T(f)\| \leq C_\varepsilon \|f\|_p + \varepsilon \, \|f\|_\Psi, \qquad \forall f \in X.$$

**Remark 5.** This theorem extends [12] Theorem II.1. As in the case of $\mathbb{C}^*$-algebras (see [3], Notes and Remarks, Chap. 15), there are miscellaneous applications of such a characterization.

**Remark 6.** Contrary to the $\Delta_2$ condition where the constant 2 may be replaced by any constant $\beta > 1$, in this $\Delta^0$ condition, the constant $\beta$ cannot be replaced by another, as the following example shows.

**Example 7.** There exists an Orlicz function $\Psi$ such that

$$\lim_{x \to +\infty} \frac{\Psi(5x)}{\Psi(x)} = +\infty, \tag{2}$$

but

$$\liminf_{x \to +\infty} \frac{\Psi(2x)}{\Psi(x)} < +\infty. \tag{3}$$

Indeed, let $(c_n)_n$ be an increasing sequence of positive numbers such that $\lim\limits_{n \to \infty} \dfrac{c_{n+1}}{c_n} = +\infty$, take $\psi(t) = c_n$ for $t \in (4^n, 4^{n+1}]$ and $\Psi(x) = \int_0^x \psi(t)\, dt$. Then (2) is verified. On the other hand, if $x_n = 2 \cdot 4^n$, one has $\Psi(x_n) \geq c_n 4^n$, and $\Psi(2x_n) \leq c_n 4^{n+1}$, so we get (3).

Before proving Theorem 4, let us note that it has the following straightforward corollary.

**Corollary 8.** *Let $X$ be like in Theorem 4, and assume that $\mathcal{F}$ is a family of operators from $X$ into a Banach space $Y$ with the following property: there exists a bounded sequence $(g_n)_n$ in $X$ such that $\lim_{n\to\infty} \|g_n\|_1 = 0$ and such that an operator $T \in \mathcal{F}$ is compact whenever*

$$\lim_{n \to \infty} \|T g_n\| = 0.$$

*Then every weakly compact operator in $T \in \mathcal{F}$ is actually compact.*

In the forthcoming paper [13], we prove, using a generalization of the notion of Carleson measure, that a composition operator $C_\phi \colon H^\Psi \to H^\Psi$

($H^\Psi$ is the space of analytic functions on the unit disk $\mathbb{D}$ of the complex plane whose boundary values are in $L^\Psi(\partial\mathbb{D})$, and $\phi\colon \mathbb{D} \to \mathbb{D}$ is an analytic self-map) is compact whenever

$$\lim_{r\to 1^-} \sup_{|\xi|=1} \Psi^{-1}\big(1/(1-r)\big)\|C_\phi(u_{\xi,r})\|_\Psi = 0,$$

where

$$u_{\xi,r}(z) = \Big(\frac{1-r}{1-\bar{\xi}rz}\Big)^2, \quad |z| < 1,$$

and we have

$$\lim_{r\to 1^-} \sup_{|\xi|=1} \Psi^{-1}\big(1/(1-r)\big)\|C_\phi(u_{\xi,r})\|_\Psi = 0$$

when $C_\phi$ is weakly compact and $\Psi \in \Delta^0$.

Though the situation does not fit exactly as in Corollary 8 (not because of the space $H^\Psi$, which is not a subspace of $M^\Psi$: we actually work in $HM^\Psi = H^\Psi \cap M^\Psi$ since $u_{\xi,r} \in HM^\Psi$, but because of the fact that we ask a uniform limit for $|\xi| = 1$), the same ideas allow us to get, when $\Psi$ satisfies the condition $\Delta^0$, that $C_\phi$ is compact if and only if it is weakly compact.

*Proof of Theorem 4.* Assume that we have (W). We may assume that $p > 1$, since if (W) is satisfied for some $p \geq 1$, it is satisfied for all $q \geq p$. Moreover, we may assume that $L^\Psi \overset{j}{\hookrightarrow} L^p$ since $\Psi$ satisfies condition $\Delta^0$ (since we have: $\lim_{x\to+\infty} \frac{\Psi(x)}{x^r} = +\infty$, for every $r > 0$). Then $T\big[(1/C_\varepsilon)B_{L^p} \cap (1/\varepsilon)B_X\big] \subseteq 2B_Y$. Taking the polar of these sets, we get $T^*(B_{Y^*}) \subseteq (2C_\varepsilon)B_{j^*[(L^p)^*]} + (2\varepsilon)B_{X^*}$, for every $\varepsilon > 0$. By a well-known lemma of Grothendieck, we get, since $B_{j^*[(L^p)^*]}$ is weakly compact, that $T^*(B_{Y^*})$ is relatively weakly compact, *i.e.* $T^*$, and hence also $T$, is weakly compact.

Conversely, assume in Theorem 4 that $T$ is weakly compact. We are going to show that (W) is satisfied with $p = 1$ (hence for all finite $p \geq 1$). Let $\varepsilon > 0$. Since the $\Delta^0$ condition implies that the complementary function of $\Psi$ satisfies the $\Delta_2$ condition, Corollary 2 implies that, when $\|f\|_\Psi = 1$

$$\|Tf\| \leq C_{\varepsilon/2} \int_\Omega \Psi\big((\varepsilon/2)|f|\big)\, d\mathbb{P} + \varepsilon/2.$$

As $\Psi$ satisfies the $\Delta^0$ condition, there is some $\beta > 1$ such that $\dfrac{\Psi(x)}{\Psi(\beta x)} \to 0$ as $x \to \infty$; hence, with $\kappa = \varepsilon/2C_{\varepsilon/2}$, there exists some $x_\kappa > 0$ such that $\Psi(x) \leq \kappa\Psi(\beta x)$ for $x \geq x_\kappa$. By the convexity of $\Psi$, one has

$\Psi(x) \leq \frac{\Psi(x_\kappa)}{x_\kappa} x =: K_\kappa x$ for $0 \leq x \leq x_\kappa$. Hence, for every $x \geq 0$ $\Psi(x) \leq \kappa \Psi(\beta x) + K_\kappa x$. It follows that, for $f \in X$, with $\|f\|_\Psi = 1$

$$\int_\Omega \Psi\big((\varepsilon/2)|f|\big)\, d\mathbb{P} \leq \kappa \int_\Omega \Psi\big(\beta(\varepsilon/2)|f|\big)\, d\mathbb{P} + K_\kappa \frac{\varepsilon}{2}\|f\|_1 \leq \kappa + K_\kappa \frac{\varepsilon}{2}\|f\|_1$$

if we have chosen $\varepsilon \leq 2/\beta$. Hence

$$\|Tf\| \leq C_{\varepsilon/2}\Big(\kappa + K_\kappa \frac{\varepsilon}{2}\|f\|_1\Big) + \frac{\varepsilon}{2} = C_{\varepsilon/2}\frac{\varepsilon}{2}K_\kappa \|f\|_1 + \Big(C_{\varepsilon/2}\kappa + \frac{\varepsilon}{2}\Big) = C'_\varepsilon \|f\|_1 + \varepsilon,$$

which is (W). □

**Remark 9.** The sufficient condition is actually a general fact, which is surely well known (see [11], Theorem 1.1, for a similar result, and [3], Theorem 15.2 for $\mathcal{C}(K)$; see also [8], page 81), and has close connection with interpolation (see [2], Proposition 1), but we have found no reference, and so we shall state it separately without proof (the proof follows that given in [3], page 310).

**Proposition 10.** *Let $T\colon X \to Y$ be an operator between two Banach spaces. Assume that there is a Banach space $Z$ and a weakly compact map $j\colon X \to Z$ such that: for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\|Tx\| \leq C_\varepsilon \|jx\|_Z + \varepsilon\, \|x\|_X\,, \quad x \in X.$$

*Then $T$ is weakly compact.*

Note that, by the Davis-Figiel-Johnson-Pełczyński factorization theorem, we may assume that $Z$ is reflexive. We may also assume that $j$ is injective, because $\ker j \subseteq \ker T$, so $T$ induces a map $\tilde{T}\colon X/\ker j \to Y$ with the same property as $T$. Indeed, if $jx = 0$, then $\|Tx\| \leq \varepsilon\|x\|$ for every $\varepsilon > 0$, and hence $Tx = 0$.

*Proof of Theorem 1.* Assume first that $T$ fixes a copy of $c_0$. There are hence some $\delta > 0$ and a sequence $(f_n)_n$ in $X$ equivalent to the canonical basis of $c_0$ such that $\|f_n\|_\Psi = 1$ and $\|Tf_n\| \geq \delta$. In particular, there is some $M > 0$ such that, for every choice of $\varepsilon_n = \pm 1$

$$\Big\| \sum_{n=1}^N \varepsilon_n f_n \Big\|_\Psi \leq M, \qquad N \geq 1.$$

Let $(r_n)_n$ be a Rademacher sequence. We have, first by Khintchine's inequality, then by Jensen's inequality and Fubini's Theorem

$$\begin{aligned}\int_\Omega \Psi\Big(\frac{1}{M\sqrt{2}}\Big(\sum_{n=1}^N |f_n|^2\Big)^{1/2}\Big)\,d\mathbb{P} &\le \int_\Omega \Psi\Big[\frac{1}{M}\int_0^1 \Big|\sum_{n=1}^N r_n(t) f_n\Big|\,dt\Big]\,d\mathbb{P}\\ &\le \int_\Omega \int_0^1 \Psi\Big[\frac{1}{M}\Big|\sum_{n=1}^N r_n(t) f_n\Big|\,dt\Big]\,d\mathbb{P}\\ &= \int_0^1 \int_\Omega \Psi\Big[\frac{1}{M}\Big|\sum_{n=1}^N r_n(t) f_n\Big|\,d\mathbb{P}\Big]\,dt \le 1.\end{aligned}$$

The monotone convergence Theorem gives then

$$\int_\Omega \Psi\Big(\frac{1}{M\sqrt{2}}\Big(\sum_{n=1}^\infty |f_n|^2\Big)^{1/2}\Big)\,d\mathbb{P} \le 1.$$

In particular, $\sum_{n=1}^\infty |f_n|^2$ is finite almost everywhere, and hence $f_n \to 0$ almost everywhere. Since $\Psi\Big(\frac{1}{M\sqrt{2}}\big(\sum_{n=1}^\infty |f_n|^2\big)^{1/2}\Big) \in L^1$, by the above inequalities, Lebesgue's dominated convergence Theorem gives

$$\int_\Omega \Psi\Big(\frac{|f_n|}{M\sqrt{2}}\Big)\,d\mathbb{P} \underset{n\to\infty}{\longrightarrow} 0.$$

But that contradicts (1) with $\varepsilon \le 1/M\sqrt{2}$ and $\varepsilon < \delta$, since $\|Tf_n\| \ge \delta$.

The converse follows from the following lemma.

**Lemma 11.** *Let $X$ be a subspace of $M^\Psi$, and let $(h_n)_n$ be a sequence in $X$, with $\|h_n\|_\Psi = 1$ for all $n \ge 1$, and such that, for some $M > 0$*

$$\int_\Omega \Psi(|h_n|/M)\,d\mathbb{P} \underset{n\to\infty}{\longrightarrow} 0.$$

*Then $(h_n)_n$ has a subsequence equivalent to the canonical basis of $c_0$.*

Indeed, if condition (1) is not satisfied, there exist some $\varepsilon_0 > 0$ and functions $h_n \in X$ with $\|h_n\|_\Psi = 1$ such that $\|Th_n\| \ge 2^n \int_\Omega \Psi(\varepsilon_0 |h_n|)\,d\mathbb{P} + \varepsilon_0$. That implies that $\int_\Omega \Psi(\varepsilon_0|h_n|)\,d\mathbb{P}$ tends to $0$, so Lemma 11 ensures that $(h_n)_n$ has a subsequence, which we shall continue to denote by $(h_n)_n$, equivalent to the canonical basis of $c_0$. Then $(Th_n)_n$ is weakly unconditionally Cauchy. Since $\|Th_n\| \ge \varepsilon_0$, $(Th_n)_n$ has, by Bessaga-Pełczyński's Theorem, a further subsequence equivalent to the canonical basis of $c_0$. It is then obvious that $T$ realizes an isomorphism between the spaces generated by these subsequences. □

*Proof of Lemma 11.* The proof uses the idea of the construction made in the proof of Theorem II.1 in [12], which it generalizes, but with some additional details.

By the continuity of $\Psi$, there exists $a > 0$ such that $\Psi(a) = 1$. Then, since $\Psi$ is increasing, we have, for every $g \in L^\infty$, $\int_\Omega \Psi\Big(a \frac{|g|}{\|g\|_\infty}\Big)\, d\mathbb{P} \leq 1$, and so $\|g\|_\Psi \leq (1/a)\, \|g\|_\infty$. Now, choose, for every $n \geq 1$, positive numbers $\alpha_n < a/2^{n+2}$ such that $\Psi(\alpha_n/2M) \leq 1$.

We are going to construct inductively a subsequence $(f_n)_n$ of $(h_n)_n$, a sequence of functions $g_n \in L^\infty$ and two sequences of positive numbers $\beta_n$ and $\varepsilon_n \leq \min\{1/2^{n+1}, M/2^{n+1}\}$, such that, for every $n \geq 1$

(i) if we set $M_1 = 1$ and, for $n \geq 2$

$$M_n = \max\Big\{1, \Psi\Big(\frac{\|g_1\|_\infty + \cdots + \|g_{n-1}\|_\infty}{2M}\Big)\Big\},$$

then $M_n \beta_n \leq 1/2^{n+1}$;

(ii) $\|f_n\|_\Psi = 1$;

(iii) $\|f_n - g_n\|_\Psi \leq \varepsilon_n$, with $\varepsilon_n$ such that $\beta_n \Psi(\alpha_n/2\varepsilon_n) \geq 2$;

(iv) $\mathbb{P}(\{|g_n| > \alpha_n\}) \leq \beta_n$;

(v) $\|\breve{g}_n\|_\Psi \geq 1/2$, with $\breve{g}_n = g_n\, \mathbb{1}_{\{|g_n|>\alpha_n\}}$.

We shall give only the inductive step, since the starting one unfolds identically. Suppose hence that the functions $f_1, \ldots, f_{n-1}$, $g_1, \ldots, g_{n-1}$ and the numbers $\beta_1, \ldots, \beta_{n-1}$ and $\varepsilon_1, \ldots, \varepsilon_{n-1}$ have been constructed. Choose then $\beta_n > 0$ such that $M_n \beta_n \leq 1/2^{n+1}$. Note that $M_n \geq 1$ implies that $\beta_n \leq 1/2^{n+1}$. Since $\int_\Omega \Psi(|h_k|/M)\, d\mathbb{P} \to 0$ as $n \to \infty$, we can find $f_n = h_{k_n}$ such that $\|f_n\|_\Psi = 1$, and moreover

$$\mathbb{P}(\{|f_n| > \alpha_n/2\}) \leq \frac{1}{\Psi(\alpha_n/2M)} \int_\Omega \Psi\Big(\frac{|f_n|}{M}\Big)\, d\mathbb{P} \leq \frac{\beta_n}{2}\cdot$$

Take now $\varepsilon_n \leq \min\{1/2^{n+1}, M/2^{n+1}\}$ such that $0 < \varepsilon_n \leq \alpha_n/2\Psi^{-1}(2/\beta_n)$ and $g_n \in L^\infty$ such that $\|f_n - g_n\|_\Psi \leq \varepsilon_n$. Then, since

$$\mathbb{P}(\{|f_n - g_n| > \alpha_n/2\})\Psi\Big(\frac{\alpha_n}{2\varepsilon_n}\Big) \leq \int_\Omega \Psi\Big(\frac{|f_n - g_n|}{\varepsilon_n}\Big)\, d\mathbb{P} \leq 1,$$

we have

$$\begin{aligned}\mathbb{P}(\{|g_n| > \alpha_n\}) &\leq \mathbb{P}(\{|f_n| > \alpha_n/2\}) + \mathbb{P}(\{|f_n - g_n| > \alpha_n/2\}) \\ &\leq \frac{\beta_n}{2} + \frac{1}{\Psi(\alpha_n/2\varepsilon_n)} \leq \beta_n.\end{aligned}$$

To end the construction, it remains to note that

$$\begin{aligned}\|f_n - \breve{g}_n\|_\Psi &\le \|f_n - g_n\|_\Psi + \|\breve{g}_n - g_n\|_\Psi \le \varepsilon_n + \frac{1}{a}\|\breve{g}_n - g_n\|_\infty \\ &\le \frac{1}{2^{n+1}} + \frac{\alpha_n}{a} \le \frac{1}{2^n} \le \frac{1}{2}\end{aligned}$$

and so

$$\|\breve{g}_n\|_\Psi \ge \|f_n\|_\Psi - \|f_n - \breve{g}_n\|_\Psi \ge 1 - \frac{1}{2} = \frac{1}{2}\cdot$$

This ends the inductive construction.

Consider now

$$\breve{g} = \sum_{n=1}^{+\infty} |\breve{g}_n| \,.$$

Set $A_n = \{|g_n| > \alpha_n\}$ and, for $n \ge 1$

$$B_n = A_n \setminus \bigcup_{j>n} A_j \,.$$

We have $\mathbb{P}\big(\limsup A_n\big) = 0$, because

$$\sum_{n\ge 1} \mathbb{P}(A_n) \le \sum_{n\ge 1} \beta_n \le \sum_{n\ge 1} \frac{1}{2^n} < +\infty.$$

Now $\breve{g}$ vanishes out of $\bigcup_{n\ge 1} B_n \cup \big(\limsup A_n\big)$ and we have

$$\begin{aligned}\int_{B_n} \Psi\Big(\frac{|\breve{g}_n|}{2M}\Big)\, d\mathbb{P} &\le \int_\Omega \Psi\Big(\frac{|g_n|}{2M}\Big)\, d\mathbb{P} \\ &\le \int_\Omega \Psi\Big(\frac{|g_n - f_n|}{2M} + \frac{|f_n|}{2M}\Big)\, d\mathbb{P} \\ &\le \frac{1}{2}\int_\Omega \Psi\Big(\frac{|g_n - f_n|}{M}\Big)\, d\mathbb{P} + \frac{1}{2}\int_\Omega \Psi\Big(\frac{|f_n|}{M}\Big)\, d\mathbb{P}.\end{aligned}$$

The first integral is less than $\varepsilon_n/M$, because $\Psi(at) \le a\Psi(t)$ for $0 \le a \le 1$ and $\varepsilon_n/M \le 1$, so that

$$\int_\Omega \Psi\Big(\frac{|g_n - f_n|}{M}\Big)\, d\mathbb{P} \le \frac{\varepsilon_n}{M}\int_\Omega \Psi\Big(\frac{|g_n - f_n|}{\varepsilon_n}\Big)\, d\mathbb{P} \le \frac{\varepsilon_n}{M} \le \frac{1}{2^{n+1}}$$

since $\|f_n - g_n\|_\Psi \le \varepsilon_n$. Since

$$\int_\Omega \Psi\Big(\frac{|f_n|}{M}\Big)\, d\mathbb{P} \le \frac{\beta_n}{2}\Psi\big(\alpha_n/2M\big) \le \beta_n/2,$$

we obtain

$$\int_{B_n} \Psi\Big(\frac{|\breve{g}_n|}{2M}\Big)\, d\mathbb{P} \leq \frac{1}{2^{n+2}} + \frac{\beta_n}{4}\,\cdot$$

Therefore, since $\mathbb{P}(B_n) \leq \mathbb{P}(A_n) \leq \beta_n$, we have

$$\begin{aligned}
\int_\Omega \Psi\Big(\frac{|\breve{g}|}{4M}\Big)\, d\mathbb{P} &= \sum_{n=1}^{+\infty} \int_{B_n} \Psi\Big(\frac{|\breve{g}|}{4M}\Big)\, d\mathbb{P} \\
&\leq \sum_{n=1}^{+\infty} \int_{B_n} \frac{1}{2}\Big[\Psi\Big(\frac{\|g_1\|_\infty + \cdots + \|g_{n-1}\|_\infty}{2M}\Big) + \Psi\Big(\frac{|\breve{g}_n|}{2M}\Big)\Big]\, d\mathbb{P} \\
&\qquad \text{by convexity of } \Psi \text{ and because } \breve{g}_j = 0 \text{ on } B_n \text{ for } j > n \\
&\leq \frac{1}{2}\sum_{n=1}^{+\infty} \Big(M_n\beta_n + \frac{1}{2^{n+2}} + \frac{\beta_n}{4}\Big) \\
&\leq \frac{1}{2}\sum_{n=1}^{+\infty} \Big(\frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+2}}\Big) \leq 1
\end{aligned}$$

which proves that $\breve{g} \in L^\Psi$, and consequently that the series $\sum_{n\geq 1} \breve{g}_n$ is weakly unconditionally Cauchy in $L^\Psi$:

$$\sup_{n\geq 1} \sup_{\theta_k = \pm 1} \Big\| \sum_{k=1}^n \theta_k \breve{g}_k \Big\|_\Psi \leq \sup_{n\geq 1} \Big\| \sum_{k=1}^n |\breve{g}_k| \Big\|_\Psi \leq \|\breve{g}\|_\Psi \leq 4M.$$

Since $\|\breve{g}_n\|_\psi \geq 1/2$, $(\breve{g}_n)_{n\geq 1}$ has, by Bessaga-Pełczyński's theorem, a subsequence $(\breve{g}_{n_k})_{k\geq 1}$ which is equivalent to the canonical basis of $c_0$. The corresponding subsequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$ remains equivalent to the canonical basis of $c_0$, since

$$\sum_{n=1}^{+\infty} \|f_n - \breve{g}_n\|_\Psi \leq \sum_{n=1}^{+\infty} \varepsilon_n + \frac{\alpha_n}{a} \leq \sum_{n=1}^{+\infty} \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} < 1$$

and the assertion follows. □

## 3. Comments

**Remark 12.** Let us note that the assumption $X \subseteq M^\Psi$ in Theorem 4 cannot be relaxed in general. In fact, suppose that $X$ is a subspace of $L^\Psi$ containing $L^\infty$, and let $\xi \in (M^\psi)^\perp \subseteq (L^\Psi)^*$, $\xi \neq 0$. Being of rank one, $\xi$ is trivially weakly compact. Suppose that it satisfies (W). Let $f \in X$ with norm 1, and let $\varepsilon > 0$. For $t$ large enough and $f_t = f\mathbb{1}_{\{|f|\leq t\}}$, we have $\|f - f_t\|_2 \leq \varepsilon / C_\varepsilon$. Moreover, $f_t \in L^\infty \subseteq X$ and $\|f_t\|_\Psi \leq \|f\|_\Psi = 1$. Since

$\xi$ vanishes on $L^\infty$ and $f - f_t \in X$, we get

$$|\xi(f)| = |\xi(f - f_t)| \leq C_\varepsilon \|f - f_t\|_2 + \varepsilon \|f - f_t\|_\Psi \leq 3\varepsilon.$$

This implies that $\xi(f) = 0$. Since this occurs for every $\xi \in (M^\Psi)^\perp$, we get that $X \subseteq M^\Psi$ (and actually $X = M^\Psi$ since $X$ contains $L^\infty$).

In particular Theorem 4 does not hold for $X = L^\Psi$.

**Remark 13.** However, condition (W) remains true for bi-adjoint operators coming from subspaces of $M^\Psi$: if $T\colon X \subseteq M^\Psi \to Y$ satisfies the condition (W), then $T^{**}\colon X^{**} \to Y^{**}$ also satisfies it. Indeed, for every $\varepsilon > 0$, we get an equivalent norm $|||\,.\,|||_\varepsilon$ on $X$ by putting

$$|||f|||_\varepsilon = C_\varepsilon \|f\|_2 + \varepsilon \|f\|_\Psi.$$

Hence if $f \in X^{**}$, there exists a net $(f_\alpha)_\alpha$ of elements in $X$, with $|||f_\alpha|||_\varepsilon \leq |||f|||_\varepsilon$ which converges weak-star to $f$. Then $(Tf_\alpha)_\alpha$ converges weak-star to $T^{**}f$, and

$$\begin{aligned} \|T^{**}f\| &\leq \liminf_\alpha \|Tf_\alpha\| \leq \liminf_\alpha (C_\varepsilon \|f_\alpha\|_2 + \varepsilon \|f_\alpha\|_\Psi) \\ &= \liminf_\alpha |||f_\alpha|||_\varepsilon \leq |||f|||_\varepsilon = C_\varepsilon \|f\|_2 + \varepsilon \|f\|_\Psi. \end{aligned}$$

Hence, from Proposition 10 above, for such a $T$, $T^{**}$ is weakly compact if and only if it satisfies (W). We shall use this fact in the forthcoming paper [13].

**Remark 14.** In Theorem 4, we cannot only assume that $\Psi \notin \Delta_2$, instead of $\Psi \in \Delta^0$, as the following example shows. It also shows that in Corollary 2, we cannot obtain condition (W) instead of condition (1).

**Example 15.** Let us define

$$\psi(t) = \begin{cases} t & \text{for} \quad 0 \leq t < 1, \\ (k!)(k+2)t - k!(k+1)! & \text{for} \quad k! \leq t \leq (k+1)!, \quad k \geq 1, \end{cases}$$

($\psi(k!) = (k!)^2$ for every integer $k \geq 1$ and $\psi$ is linear between $k!$ and $(k+1)!$), and $\Psi(x) = \int_0^x \psi(t)\,dt$. Since $t^2 \leq \psi(t)$ for all $t \geq 0$, one has $x^3/3 \leq \Psi(x)$ for all $x \geq 0$. Then

$$\Psi(2.n!) \geq \int_{n!}^{2.n!} \psi(t)\,dt = n!(n+2)\frac{3}{2}(n!)^2 - (n!)^2(n+1)! = (n!)^3\Big(\frac{n}{2} + 2\Big),$$

whereas

$$\Psi(n!) = \int_0^{n!} \psi(t)\,dt \leq (n!)^2\, n! = (n!)^3\,;$$

hence

$$\frac{\Psi(2.n!)}{\Psi(n!)} \geq \frac{n}{2} + 2,$$

and so

$$\limsup_{x\to+\infty} \frac{\Psi(2x)}{\Psi(x)} = +\infty,$$

which means that $\Psi \notin \Delta_2$. On the other hand, for every $\beta > 1$

$$\Psi(n!/\beta) \geq \frac{1}{3}\Big(\frac{n!}{\beta}\Big)^3 = \frac{(n!)^3}{3\beta^3}\,,$$

so

$$\frac{\Psi(n!)}{\Psi(n!/\beta)} \leq \frac{(n!)^3}{(n!)^3/3\beta^3} = 3\beta^3\,;$$

hence

$$\liminf_{x\to+\infty} \frac{\Psi(2x)}{\Psi(x)} \leq 3\beta^3\,,$$

and $\Psi \notin \Delta^0$ (actually, this will follow too from the fact that Theorem 4 is not valid for this $\Psi$).

Moreover, the conjugate function of $\Psi$ satisfies the condition $\Delta_2$. Indeed, since $\psi$ is convex, one has $\psi(2u) \geq 2\psi(u)$ for all $u \geq 0$, and hence:

$$\Psi(2x) = \int_0^{2x} \psi(t)\,dt = 2\int_0^x \psi(2u)\,du \geq 2\int_0^x 2\psi(u)\,du = 4\Psi(x),$$

and as it was seen in the Introduction that means that $\Psi \in \nabla_2$.

Now, we have $x^3/3 \leq \Psi(x)$ for all $x \geq 0$; therefore $\|\,.\,\|_3 \leq 3^{1/3}\|\,.\,\|_\Psi$. In particular, we have an inclusion map $j\colon M^\Psi \hookrightarrow L^3$, which is, of course, weakly compact. Nevertheless, assuming that $\mathbb{P}$ is diffuse, condition (W) is not verified by $j$, when $\varepsilon < 1$. Indeed, as we have seen before, one has $\Psi(n!) \leq (n!)^3$. Hence, if we choose a measurable set $A_n$ such that $\mathbb{P}(A_n) = 1/\Psi(n!)$, we have

$$\|\mathbb{1}_{A_n}\|_\Psi = \frac{1}{\Psi^{-1}\big(1/\mathbb{P}(A_n)\big)} = \frac{1}{n!}\,;$$

whereas

$$\|\mathbb{1}_{A_n}\|_3 = \mathbb{P}(A_n)^{1/3} = \frac{1}{\Psi(n!)^{1/3}} \geq \frac{1}{n!}$$

and

$$\|\mathbb{1}_{A_n}\|_2 = \mathbb{P}(A_n)^{1/2} \leq \left[\frac{3}{(n!)^3}\right]^{1/2} = \frac{\sqrt{3}}{(n!)^{3/2}}\,.$$

If condition (W) were true, we should have, for every $n \geq 1$

$$\frac{1}{n!} \leq C_\varepsilon \frac{\sqrt{3}}{(n!)^{3/2}} + \varepsilon \frac{1}{n!}\,,$$

that is $\sqrt{n!} \leq \sqrt{3}\, \frac{C_\varepsilon}{1-\varepsilon}$, which is of course impossible for $n$ large enough.

**Remark 16.** In the case of the whole space $M^\Psi$, we can give a direct proof of the necessity in Theorem 4. Indeed, suppose that $T\colon M^\Psi \to X$ is weakly compact. Then $T^*\colon X^* \to L^\Phi = (M^\Psi)^*$ is weakly compact, and so the set $K = T^*(B_{X^*})$ is relatively weakly compact.

Since $\Phi$ satisfies the $\Delta^0$ condition, it follows from [1] (Corollary 2.9) that $K$ has equi-absolutely continuous norms. Hence, for every $\varepsilon > 0$, we can find $\delta_\varepsilon > 0$ such that:

$$m(A) \leq \delta_\varepsilon \qquad \Rightarrow \qquad \|g\mathbb{1}_A\|_\Phi \leq \varepsilon/2\,, \quad g \in T^*(B_{X^*}).$$

But (the factor $1/2$ appears because we use the Luxemburg norm on the dual, and not the Orlicz norm: see [16], Proposition III.3.4)

$$\begin{aligned}\sup_{g\in T^*(B_{X^*})} \|g\mathbb{1}_A\|_\Phi &\geq \frac{1}{2} \sup_{u\in B_{X^*}} \sup_{\|f\|_\Psi\leq 1} |< f, (T^*u)\mathbb{1}_A >| \\ &= \frac{1}{2} \sup_{u\in B_{X^*}} \sup_{\|f\|_\Psi\leq 1} \Big| \int f(T^*u)\mathbb{1}_A \, dm\Big| \\ &= \frac{1}{2} \sup_{u\in B_{X^*}} \sup_{\|f\|_\Psi\leq 1} |< T(f\mathbb{1}_A), u >| = \frac{1}{2} \sup_{\|f\|_\Psi\leq 1} \|T(f\mathbb{1}_A)\|;\end{aligned}$$

so

$$m(A) \leq \delta_\varepsilon \qquad \Rightarrow \qquad \sup_{\|f\|_\Psi\leq 1} \|T(f\mathbb{1}_A)\| \leq \varepsilon.$$

Now, we have

$$m(|f| \geq \|f\|_2/\delta_\varepsilon) \leq \frac{\delta_\varepsilon}{\|f\|_2} \int |f|\, dm = \frac{\delta_\varepsilon}{\|f\|_2} \|f\|_1 \leq \delta_\varepsilon;$$

hence, with $A = \{|f| \geq \|f\|_2/\delta_\varepsilon\}$, we get, for $\|f\|_\Psi \leq 1$:

$$\|Tf\| \leq \|T(f\mathbb{1}_A)\| + \|T(f\mathbb{1}_{A^c})\| \leq \varepsilon + \|T\| \frac{\|f\|_2}{\delta_\varepsilon}$$

since $|f\mathbb{1}_{A^c}| \leq \|f\|_2/\delta_\varepsilon$ implies $\|f\mathbb{1}_{A^c}\|_\Psi \leq \|f\mathbb{1}_{A^c}\|_\infty \leq \|f\|_2/\delta_\varepsilon$.

**Remark 17.** Conversely, E. Lavergne ([10]) recently uses our Theorem 4 to give a proof of the above quoted result of J. Alexopoulos ([1], Corollary 2.9), and uses it to show that, when $\Psi \in \Delta^0$, then the reflexive subspaces of $L^{\Phi}$ (where $\Phi$ is the conjugate of $\Psi$) are closed for the $L^1$-norm.

*Acknowledgements.* This work was made during the stay in Lens as *Professeur invité de l'Université d'Artois*, in May–June 2005, and a visit in Lens and Lille in March 2006 of the fourth-named author. He would like to thank both Mathematics Departments for their kind hospitality.

## References

[1] J. Alexopoulos, *De la Vallée–Poussin's theorem and weakly compact sets in Orlicz spaces*, Quaestiones Math., 17 (1994), 231–248.

[2] B. Beauzamy, *Propriétés géométriques des espaces d'interpolation, Séminaire Maurey-Schwartz* 1974–1975: Espaces $L_p$, applications radonifiantes et géométrie des espaces de Banach, Exp. No. XIV, 17 pp., Centre Math., École Polytech., Paris, 1975.

[3] J. Diestel, H. Jarchow and A. Tonge, *Absolutely Summing Operators*, Cambridge studies in advanced mathematics, 43, Cambridge University Press, 1995.

[4] G. Godefroy and P. Saab, *Quelques espaces de Banach ayant les propriétés* $(V)$ *ou* $(V^*)$ *de A. Pełczyński*, C.R. Acad. Sci. Paris, Série A, 303 (1986), 503–506.

[5] G. Godefroy and P. Saab, *Weakly unconditionally convergent series in* $M$*-ideals*, Math. Scand., 64 (1989), 307–318.

[6] P. Harmand, D. Werner and W. Werner, $M$*-ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Math., 1547, Springer, 1993.

[7] H. Jarchow, *On Weakly Compact Operators on* $\mathbf{C}^*$*-Algebras*, Math. Ann., 273 (1986), 341–343.

[8] H. Jarchow and U. Matter, *On weakly compact operators on* $\mathcal{C}(K)$*-spaces*, 80–88, Proceedings of the Missouri Conference held in Columbia, USA, June 24–29 1984, N. Kalton and E. Saab, Eds., Lecture Notes in Math., 1166, 1985.

[9] M. A. Krasnosel'skiĭ and Ya. B. Rutickiĭ, *Convex functions and Orlicz spaces (translation)*, P. Noordhoff Ltd., Groningen, 1961.

[10] E. Lavergne, *Reflexive subspaces of some Orlicz spaces*, Colloq. Math., 113 (2008), 333–340.

[11] P. Lefèvre, *Some characterizations of weakly compact operators on* $H^\infty$ *and on the disk algebra. Application to composition operators*, Journal of Operator Theory, 54 (2005), 229–238.

[12] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, *Some translation-invariant Banach function spaces which contain* $c_0$, Studia Math., 163 (2004), 137–155.

[13] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, *Composition operators on Hardy-Orlicz spaces*, Memoirs Amer. Math. Soc., to appear.

[14] C. P. Niculescu, *Absolute continuity and weak compactness*, Bull. Amer. Math. Soc., 81 (1975), 1064–1066.

[15] C. P. Niculescu, *Absolute continuity in Banach space theory*, Rev. Roum. Math. Pures Appl., 24 (1979), 413–422.

[16] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Pure and Applied Mathematics 146, Marcel Dekker, Inc., 1991.

[17] D. Werner, *New classes of Banach spaces which are* $M$*-ideals in their biduals*, Math. Proc. Cambridge Philos. Soc., 111 (1992), 337–354.

P. Lefèvre and D. Li
Université d'Artois
Laboratoire de Mathématiques de Lens EA 2462
Fédération CNRS Nord-Pas-de-Calais FR 2956
Faculté des Sciences Jean Perrin, Rue Jean Souvraz, S.P.18
62 307 LENS Cedex
FRANCE
(*E-mail : pascal.lefevre@euler.univ-artois.fr*)
(*E-mail : daniel.li@euler.univ-artois.fr*)

H. Queffélec
Université des Sciences et Technologies de Lille
Laboratoire Paul Painlevé U.M.R. CNRS 8524
U.F.R. de Mathématiques,
59 655 VILLENEUVE D'ASCQ Cedex
FRANCE
(*E-mail : queff@math.univ-lille1.fr*)

Luis Rodríguez-Piazza
Universidad de Sevilla
Facultad de Matemáticas, Dpto de Análisis Matemático
Apartado de Correos 1160
41 080 SEVILLA
SPAIN
(*E-mail : piazza@us.es*)

(*Received : September 2007*)