Gilles Godefroy, ℓ_1, et moi; et d'autres ...

Gilles Godefroy, ℓ_1, and I; and some others ...

Daniel Li

Université d'Artois (Lens, France)

Analysis Meeting on the occasion of the 60th birthday of our colleague and friend Gilles Godefroy

Mons - November 4–5, 2013


ROBERT LAMOUREUX
GABY MORLAY
FERNAND LEDOUX
NICOLE COURCEL
JEAN-PAUL LE CHANOIS
PAPA, MAMAN la BONNE et MOI
LOUIS DE FUNES
JEAN TISSIER
DVD
COLLECTIONNEUR

A movie from 1954

A movie from $1954 \approx 1953 + \ell_1$!!!!


Joyeux
Anniversaire !

Gilles Godefroy and ℓ_1

Theorem [Gilles (1988)]

A separable Banach space X contains ℓ_1 if and only if there exists an equivalent norm $|||\,.\,|||$ on X and $z \in X^{**} \setminus \{0\}$ such that :

$$|||x + z||| = |||x||| + |||z||| \qquad \forall x \in X.$$

Theorem [Gilles (1988)]

A separable Banach space X contains ℓ_1 if and only if there exists an equivalent norm $|||\,.\,|||$ on X and $z \in X^{**} \setminus \{0\}$ such that :

$$|||x + z||| = |||x||| + |||z||| \qquad \forall x \in X.$$

Sufficient condition: easy, by local reflexivity and induction.

Gilles Godefroy and ℓ_1: One of my favourite Gilles's result

Theorem [Gilles (1988)]

A separable Banach space X contains ℓ_1 if and only if there exists an equivalent norm $|||\,.\,|||$ on X and $z \in X^{**} \setminus \{0\}$ such that :

$$|||x + z||| = |||x||| + |||z||| \qquad \forall x \in X.$$

Sufficient condition: easy, by local reflexivity and induction.

Now: ball topology !

Gilles Godefroy and ℓ_1: One of my favourite Gilles's result

Theorem [Gilles (1988)]

A separable Banach space X contains ℓ_1 if and only if there exists an equivalent norm $|||\,.\,|||$ on X and $z \in X^{**} \setminus \{0\}$ such that :

$$|||x + z||| = |||x||| + |||z||| \qquad \forall x \in X.$$

Sufficient condition: easy, by local reflexivity and induction.

Now: ball topology !

It is the coarsest topology b_X on X for which all closed balls of X are closed.

The previous result gives the necessary part of:

Theorem [Gilles + N. Kalton (1989)]

A Banach space X contains ℓ_1 if and only if, on the unit ball B_X, the ball topology b_X is irreducible.

The previous result gives the necessary part of:

Theorem [Gilles + N. Kalton (1989)]

A Banach space X contains ℓ_1 if and only if, on the unit ball B_X, the ball topology b_X is irreducible.

A topological space is irreducible if two arbitrary non-empty open sets have non-empty intersection.
Equivalently here: if B_X is contained in the union of a finite number of closed balls, then B_X is contained in one of them.

The previous result gives the necessary part of:

Theorem [Gilles + N. Kalton (1989)]

A Banach space X contains ℓ_1 if and only if, on the unit ball B_X, the ball topology b_X is irreducible.

A topological space is irreducible if two arbitrary non-empty open sets have non-empty intersection.
Equivalently here: if B_X is contained in the union of a finite number of closed balls, then B_X is contained in one of them.
Such a topology is highly non-Hausdorff.

The previous result gives the necessary part of:

Theorem [Gilles + N. Kalton (1989)]

A Banach space X contains ℓ_1 if and only if, on the unit ball B_X, the ball topology b_X is irreducible.

A topological space is irreducible if two arbitrary non-empty open sets have non-empty intersection.
Equivalently here: if B_X is contained in the union of a finite number of closed balls, then B_X is contained in one of them.
Such a topology is highly non-Hausdorff.

The sufficient part is given by:

Theorem [Gilles + N. Kalton (1989)]

$$X \not\supseteq \ell_1 \quad \Longrightarrow \quad b_X \text{ is Hausdorff on } B_X.$$

Gilles Godefroy, ℓ_1 and I: two questions

In fall of 1989, Gilles asked me two questions:

In fall of 1989, Gilles asked me two questions:

Question 1

Let X be an isomorphic predual of ℓ_1 with Pełczyński's property (u); is X isomorphic to c_0?

Gilles Godefroy, ℓ_1 and I: two questions

In fall of 1989, Gilles asked me two questions:

Question 1

Let X be an isomorphic predual of ℓ_1 with Pełczyński's property (u); is X isomorphic to c_0?

Question 2

Let X be a subspace of L^1 whose unit ball B_X is compact and locally convex for the convergence in measure. Is X isomorphic to a subspace of ℓ_1?

Gilles Godefroy, ℓ_1 and I: two questions

In fall of 1989, Gilles asked me two questions:

Question 1

Let X be an isomorphic predual of ℓ_1 with Pełczyński's property (u); is X isomorphic to c_0?

Question 2

Let X be a subspace of L^1 whose unit ball B_X is compact and locally convex for the convergence in measure. Is X isomorphic to a subspace of ℓ_1?

Question 2: answered in 1994 (Gilles + N. Kalton + D. Li, published in 1996)

Gilles Godefroy, ℓ_1 and I: two questions

In fall of 1989, Gilles asked me two questions:

Question 1

Let X be an isomorphic predual of ℓ_1 with Pełczyński's property (u); is X isomorphic to c_0?

Question 2

Let X be a subspace of L^1 whose unit ball B_X is compact and locally convex for the convergence in measure. Is X isomorphic to a subspace of ℓ_1?

Question 2: answered in 1994 (Gilles + N. Kalton + D. Li, published in 1996)

Question 1: still open.

About Question 2

About Question 2

It is well known that the topology of convergence in measure τ_m is not locally convex on L^1, and does not fit well with the weak topology (there are nets in the unit ball which converge weakly to 0 but to $\mathbb{1}$ in measure).

About Question 2

It is well known that the topology of convergence in measure τ_m is not locally convex on L^1, and does not fit well with the weak topology (there are nets in the unit ball which converge weakly to 0 but to $\mathbb{1}$ in measure).

On the other hand (Kadets + Pełczyński, 1962):

$$X \subseteq L^1 \text{ is reflexive} \qquad \Longleftrightarrow \qquad \tau_m = \text{norm topology on whole } X.$$

In particular, τ_m is locally convex on X and finer than the weak topology.

It is well known that the topology of convergence in measure τ_m is not locally convex on L^1, and does not fit well with the weak topology (there are nets in the unit ball which converge weakly to 0 but to $\mathbb{1}$ in measure).

On the other hand (Kadets + Pełczyński, 1962):

$$X \subseteq L^1 \text{ is reflexive} \qquad \Longleftrightarrow \qquad \tau_m = \text{norm topology on whole } X.$$

In particular, τ_m is locally convex on X and finer than the weak topology.

Now (as easy too see): if Σ is the σ-algebra generated by a measurable partition, then $X = \mathbb{E}^{\Sigma}(L^1)$ is isometric to ℓ_1 and on its unit ball B_X, τ_m is equal to the w^*-topology of ℓ_1, so this unit ball is compact and locally convex for τ_m (the same is true for the subspaces Y of X whose unit ball is closed in measure), and weaker than the weak topology.

About Question 2

It is well known that the topology of convergence in measure τ_m is not locally convex on L^1, and does not fit well with the weak topology (there are nets in the unit ball which converge weakly to 0 but to $\mathbb{I}$ in measure).

On the other hand (Kadets + Pełczyński, 1962):

$$X \subseteq L^1 \text{ is reflexive} \qquad \Longleftrightarrow \qquad \tau_m = \text{norm topology on whole } X.$$

In particular, τ_m is locally convex on X and finer than the weak topology.

Now (as easy too see): if Σ is the σ-algebra generated by a measurable partition, then $X = \mathbb{E}^{\Sigma}(L^1)$ is isometric to ℓ_1 and on its unit ball B_X, τ_m is equal to the w^*-topology of ℓ_1, so this unit ball is compact and locally convex for τ_m (the same is true for the subspaces Y of X whose unit ball is closed in measure), and weaker than the weak topology.

What about the converse?

About Question 2

Theorem [Gilles + N. Kalton + D. Li (1996)]

Let X be a subspace of L^1 with (AP). Then its unit ball B_X is compact and locally convex for τ_m iff for every $\varepsilon > 0$ there is a w^*-closed subspace X_ε of ℓ_1 such that $\mathrm{dist}\,(X, X_\varepsilon) \leq 1 + \varepsilon$.

Theorem [Gilles + N. Kalton + D. Li (1996)]

Let X be a subspace of L^1 with (AP). Then its unit ball B_X is compact and locally convex for τ_m iff for every $\varepsilon > 0$ there is a w^*-closed subspace X_ε of ℓ_1 such that $\operatorname{dist}(X, X_\varepsilon) \leq 1 + \varepsilon$.

Main tool:

Theorem [N. Kalton + D. Werner (1995)]

Let Y be a Banach space with separable dual Y^*. If Y has property (m_1^*):

$$y_n^* \xrightarrow{w^*} 0 \qquad \Longrightarrow$$
$$\left[\limsup \|y^* + y_n^*\| = \|y^*\| + \limsup \|y_n^*\|, \forall y^* \in Y^*\right],$$

then, for all $\varepsilon > 0$, there exists a subspace Y_ε of c_0 such that $\operatorname{dist}(Y, Y_\varepsilon) \leq 1 + \varepsilon$.

About Question 2

In our case, if B_X is compact and locally convex for τ_m, then

$$X^\sharp = \{\varphi \in X^* \, ; \; \varphi_{|B_X} \tau_m\text{-}continuous\}$$

satisfies $(X^\sharp)^* = X$ and has (m_1^*).

About Question 2

In our case, if B_X is compact and locally convex for τ_m, then

$$X^\sharp = \{\varphi \in X^* \,;\; \varphi_{|B_X} \tau_m\text{-}continuous\}$$

satisfies $(X^\sharp)^* = X$ and has (m_1^*).

We used then:

Theorem [Gilles+ N. Kalton + D. Li (1996)]

Let Z be a subspace of c_0 with (AP). If Z^* is isometric to a subspace of L^1, then, for every $\varepsilon > 0$, there exists a w^*-closed subspace Y_ε of ℓ_1 such that $\mathrm{dist}\,(Z^*, Y_\varepsilon) \leq 1 + \varepsilon$.

About Question 2

In our case, if B_X is compact and locally convex for τ_m, then

$$X^\sharp = \{\varphi \in X^* \, ; \; \varphi_{|B_X} \tau_m\text{-}continuous\}$$

satisfies $(X^\sharp)^* = X$ and has (m_1^*).

We used then:

Theorem [Gilles+ N. Kalton + D. Li (1996)]

Let Z be a subspace of c_0 with (AP). If Z^* is isometric to a subspace of L^1, then, for every $\varepsilon > 0$, there exists a w^*-closed subspace Y_ε of ℓ_1 such that $\mathrm{dist}\,(Z^*, Y_\varepsilon) \leq 1 + \varepsilon$.

Uses a skipped blocking argument and a average argument, using the cotype 2 of L^1, *via* Khintchine's inequalities.

About Question 2

In our case, if B_X is compact and locally convex for τ_m, then

$$X^\sharp = \{\varphi \in X^* \, ; \; \varphi_{|B_X} \tau_m\text{-}continuous\}$$

satisfies $(X^\sharp)^* = X$ and has (m_1^*).

We used then:

Theorem [Gilles+ N. Kalton + D. Li (1996)]

Let Z be a subspace of c_0 with (AP). If Z^* is isometric to a subspace of L^1, then, for every $\varepsilon > 0$, there exists a w^*-closed subspace Y_ε of ℓ_1 such that $\mathrm{dist}\,(Z^*, Y_\varepsilon) \leq 1 + \varepsilon$.

Uses a skipped blocking argument and a average argument, using the cotype 2 of L^1, *via* Khintchine's inequalities.

This is a partial converse of a result of D. Alspach (1979): every quotient of c_0 is almost isometric to subspaces of c_0.

About Question 2: counterexample

However:

Examples [Gilles + N. Kalton + D. Li (1996 and 2000)]

1) There exists a subspace X_0 of L^1 whose unit ball is compact, but not locally convex in measure.

2) There exists a subspace X_1 of L^1 whose unit ball is compact, and locally convex in measure, but for every sub-σ-algebra Σ generated by a measurable partition, one has:

$$\sup_{f \in B_{X_1}} \|\mathbb{E}^{\Sigma} f - f\|_1 \geq 1 \quad .$$

About Question 2: counterexample

However:

Examples [Gilles + N. Kalton + D. Li (1996 and 2000)]

1) There exists a subspace X_0 of L^1 whose unit ball is compact, but not locally convex in measure.

2) There exists a subspace X_1 of L^1 whose unit ball is compact, and locally convex in measure, but for every sub-σ-algebra Σ generated by a measurable partition, one has:

$$\sup_{f \in B_{X_1}} \|\mathbb{E}^{\Sigma} f - f\|_1 \geq 1 \quad .$$

Question. If X is a subspace of L^1 almost isometric to w^*-closed subspaces of ℓ_1, does it exist a σ-algebra Σ, generated by a measurable partition such that, for every $\varepsilon > 0$, one has $d_{\tau_m}(f, \mathbb{E}^{\Sigma} f) \leq \varepsilon$ for every $f \in B_X$?

About Question 2: counterexample

However:

Examples [Gilles + N. Kalton + D. Li (1996 and 2000)]

1) There exists a subspace X_0 of L^1 whose unit ball is compact, but not locally convex in measure.

2) There exists a subspace X_1 of L^1 whose unit ball is compact, and locally convex in measure, but for every sub-σ-algebra Σ generated by a measurable partition, one has:

$$\sup_{f\in B_{X_1}} \|\mathbb{E}^{\Sigma} f - f\|_1 \geq 1 \quad .$$

Question. If X is a subspace of L^1 almost isometric to w^*-closed subspaces of ℓ_1, does it exist a σ-algebra Σ, generated by a measurable partition such that, for every $\varepsilon > 0$, one has $d_{\tau_m}(f, \mathbb{E}^{\Sigma} f) \leq \varepsilon$ for every $f \in B_X$?

In the 1996's paper, we asserted that the answer is "yes", but there is a gap in the proof.

About Question 1: strange preduals of ℓ_1

About Question 1: strange preduals of ℓ_1

Isomorphic preduals of ℓ_1 may have strange properties. In 1980, Bourgain and Delbaen gave a method to construct such peculiar $\mathcal{L}^\infty$ spaces. This construction has been used and improved by several people.

About Question 1: strange preduals of ℓ_1

Isomorphic preduals of ℓ_1 may have strange properties. In 1980, Bourgain and Delbaen gave a method to construct such peculiar $\mathcal{L}^\infty$ spaces. This construction has been used and improved by several people. For example:

Bourgain-Delbaen's spaces

1) (Bourgain + Delbaen, 1980) There are isomorphic preduals of ℓ_1 with the Radon-Nikodým property and are somewhat reflexive (every infinite dimensional subspace contains another one which is reflexive and infinite dimensional).

About Question 1: strange preduals of ℓ_1

Isomorphic preduals of ℓ_1 may have strange properties. In 1980, Bourgain and Delbaen gave a method to construct such peculiar $\mathcal{L}^\infty$ spaces. This construction has been used and improved by several people. For example:

Bourgain-Delbaen's spaces

1) (Bourgain + Delbaen, 1980) There are isomorphic preduals of ℓ_1 with the Radon-Nikodým property and are somewhat reflexive (every infinite dimensional subspace contains another one which is reflexive and infinite dimensional).

2) (Alspach, 2000) These spaces have Szlenk index equal to ω.

About Question 1: strange preduals of ℓ_1

Isomorphic preduals of ℓ_1 may have strange properties. In 1980, Bourgain and Delbaen gave a method to construct such peculiar $\mathcal{L}^\infty$ spaces. This construction has been used and improved by several people. For example:

Bourgain-Delbaen's spaces

1) (Bourgain + Delbaen, 1980) There are isomorphic preduals of ℓ_1 with the Radon-Nikodým property and are somewhat reflexive (every infinite dimensional subspace contains another one which is reflexive and infinite dimensional).

2) (Alspach, 2000) These spaces have Szlenk index equal to ω.

3) (Haydon, 2000) If X is such a space, there is some $r \in (1, \infty)$ such that every infinite dimensional subspace of X contains ℓ_r.

About Question 1: strange preduals of ℓ_1

Isomorphic preduals of ℓ_1 may have strange properties. In 1980, Bourgain and Delbaen gave a method to construct such peculiar $\mathcal{L}^\infty$ spaces. This construction has been used and improved by several people. For example:

Bourgain-Delbaen's spaces

1) (Bourgain + Delbaen, 1980) There are isomorphic preduals of ℓ_1 with the Radon-Nikodým property and are somewhat reflexive (every infinite dimensional subspace contains another one which is reflexive and infinite dimensional).

2) (Alspach, 2000) These spaces have Szlenk index equal to ω.

3) (Haydon, 2000) If X is such a space, there is some $r \in (1, \infty)$ such that every infinite dimensional subspace of X contains ℓ_r.

It won't define the Szlenk index, and I only say that the Szlenk index is an ordinal and that of c_0 is ω, the first infinite ordinal.

About Question 1: strange preduals of ℓ_1

By mixing Bourgain-Delbaen's construction with Gowers-Maurey's one:

Theorem [Argyros + Haydon (2011)]

There is an isomorphic predual of ℓ_1 which is HI (hereditarily indecomposable) and has very few operators: every operator has the form $\lambda \, \mathrm{Id} + K$, where λ is a scalar and K a compact operator.

About Question 1: strange preduals of ℓ_1

By mixing Bourgain-Delbaen's construction with Gowers-Maurey's one:

Theorem [Argyros + Haydon (2011)]

There is an isomorphic predual of ℓ_1 which is HI (hereditarily indecomposable) and has very few operators: every operator has the form $\lambda\,\mathrm{Id} + K$, where λ is a scalar and K a compact operator.

Very recently, M. Tarbard (arXiv), a student of R. Haydon, has given new examples in his thesis. For example:
1) for every $k \geq 1$, there are HI ℓ_1 preduals X_k whose Calkin algebra $\mathcal{L}(X_k)/\mathcal{K}(X_k)$ is of dimension k (for $k \geq 2$, they have few operators, but not very few);
2) there is a predual of ℓ_1 whose Calkin algebra is isometric, as a Banach algebra, to ℓ_1; consequently, it is indecomposable but not hereditarily.

One more result.

Theorem [Daws + Haydon + Schlumprecht + White (2012)]

There is an isomorphic predual F of $\ell_1(\mathbb{Z})$ which is shift-invariant and whose Szlenk index is equal to ω^2.

About Question 1: strange preduals of ℓ_1

One more result.

Theorem [Daws + Haydon + Schlumprecht + White (2012)]

There is an isomorphic predual F of $\ell_1(\mathbb{Z})$ which is shift-invariant and whose Szlenk index is equal to ω^2.

That F is shift-invariant means that F, isomorphically identified with a subspace of $\ell_\infty(\mathbb{Z})$, is invariant under the bilateral shift on $\ell_\infty(\mathbb{Z})$. Equivalently, $\ell_1(\mathbb{Z})$ is a dual Banach algebra for $\sigma(F^*, F)$.

Actually, isomorphic preduals of ℓ_1 have universal properties.

Actually, isomorphic preduals of ℓ_1 have universal properties.

Theorem [Freeman + Odell + Schlumprecht (2011)]

Every Banach space with separable dual embeds into an isomorphic predual of ℓ_1.

Actually, isomorphic preduals of ℓ_1 have universal properties.

Theorem [Freeman + Odell + Schlumprecht (2011)]

Every Banach space with separable dual embeds into an isomorphic predual of ℓ_1.

Theorem [Argyros + Freeman + Haydon + Odell + Raikoftsalis + Schlumprecht + Zisimopoulou (2012)]

Every separable reflexive Banach space X with Szlenk index ω (in particular, every uniformly convex space) embeds into an isomorphic predual Z of ℓ_1 with very few operators.

About Question 1: motivation

Question

How to distinguish c_0 among the other preduals of ℓ_1 ?

About Question 1: motivation

Question

How to distinguish c_0 among the other preduals of ℓ_1 ?

It turns out that c_0 can be characterized by unconditionality.

Question

How to distinguish c_0 among the other preduals of ℓ_1 ?

It turns out that c_0 can be characterized by unconditionality.

Indeed:

Theorem c_0

Let X be an isomorphic predual of ℓ_1; then X is isomorphic to c_0 in the following cases:

Question

How to distinguish c_0 among the other preduals of ℓ_1 ?

It turns out that c_0 can be characterized by unconditionality.

Indeed:

Theorem c_0

Let X be an isomorphic predual of ℓ_1; then X is isomorphic to c_0 in the following cases:

1) (Johnson + Zippin, 1972) if X embeds into a quotient of c_0;

Question

How to distinguish c_0 among the other preduals of ℓ_1 ?

It turns out that c_0 can be characterized by unconditionality.

Indeed:

Theorem c_0

Let X be an isomorphic predual of ℓ_1; then X is isomorphic to c_0 in the following cases:

1) (Johnson + Zippin, 1972) if X embeds into a quotient of c_0;

2) (Rosenthal, 1983) if X embeds into a space with an unconditional basis;

Question

How to distinguish c_0 among the other preduals of ℓ_1 ?

It turns out that c_0 can be characterized by unconditionality.

Indeed:

Theorem c_0

Let X be an isomorphic predual of ℓ_1; then X is isomorphic to c_0 in the following cases:

1) (Johnson + Zippin, 1972) if X embeds into a quotient of c_0;

2) (Rosenthal, 1983) if X embeds into a space with an unconditional basis;

3) (Ghoussoub + Johnson, 1989) if X embeds into an order continuous Banach lattice.

About Question 1: motivation

Note that Johnson and Zippin also proved (1974) that:

Every quotient of c_0 is isomorphic to a subspace of c_0.

In particular, every quotient of c_0 embeds into a space with an unconditional basis.

Of course, every space with an unconditional basis is an order continuous Banach lattice.

About Question 1: motivation

Note that Johnson and Zippin also proved (1974) that:

Every quotient of c_0 is isomorphic to a subspace of c_0.

In particular, every quotient of c_0 embeds into a space with an unconditional basis.

Of course, every space with an unconditional basis is an order continuous Banach lattice.

Rosenthal used a skipped blocking argument and the Grothendieck theorem (every operator $T\colon \ell_1 \to \ell_2$ is absolutely summing); unconditionality is used to glue the blocks together.

Ghoussoub and Johnson used a localized version of Rosenthal's arguments.

About Question1: Property (u)

Property (u) is a weak form of unconditionality.

Property (u) is a weak form of unconditionality.

Definition

A Banach space space X has Pełczyński's property (u) if for every weak Cauchy sequence $(x_n)_{n\geq 1}$ in X, there is a weakly unconditional Cauchy series $\sum_{n\geq 1} u_n$ such that $x_n - \sum_{k=1}^n u_k \xrightarrow[n\to\infty]{w} 0$.

Property (u) is a weak form of unconditionality.

Definition

A Banach space space X has Pełczyński's property (u) if for every weak Cauchy sequence $(x_n)_{n\geq 1}$ in X, there is a weakly unconditional Cauchy series $\sum_{n\geq 1} u_n$ such that $x_n - \sum_{k=1}^n u_k \xrightarrow[n\to\infty]{w} 0$.

Then there is a least $K \geq 1$ such that, for every Baire-1 element $x^{**} \in X^{**}$, one can choose, for every $\varepsilon > 0$, a wuC series $\sum_{n\geq 1} u_n$ such that $x^{**} = w^* - \sum_{n=1}^\infty u_n$ and:

$$\sup_{\theta_k = \pm 1, n\geq 1} \left\| \sum_{k=1}^n \theta_k u_k \right\| \leq (K+\varepsilon) \|x^{**}\| \quad ,$$

called the constant of property (u) of X.

Property (u) is a weak form of unconditionality.

Definition

A Banach space space X has Pełczyński's property (u) if for every weak Cauchy sequence $(x_n)_{n\geq 1}$ in X, there is a weakly unconditional Cauchy series $\sum_{n\geq 1} u_n$ such that $x_n - \sum_{k=1}^{n} u_k \xrightarrow[n\to\infty]{w} 0$.

Every subspace of a space with property (u) also has property (u).

Property (u) is a weak form of unconditionality.

Definition

A Banach space space X has Pełczyński's property (u) if for every weak Cauchy sequence $(x_n)_{n\geq 1}$ in X, there is a weakly unconditional Cauchy series $\sum_{n\geq 1} u_n$ such that $x_n - \sum_{k=1}^{n} u_k \xrightarrow[n\to\infty]{w} 0$.

Every subspace of a space with property (u) also has property (u).

Theorem [Tzafriri (1972)]

Every subspace of an order continuous Banach lattice has property (u).

About Question1: Property (u)

Property (u) is a weak form of unconditionality.

Definition

A Banach space space X has Pełczyński's property (u) if for every weak Cauchy sequence $(x_n)_{n\geq 1}$ in X, there is a weakly unconditional Cauchy series $\sum_{n\geq 1} u_n$ such that $x_n - \sum_{k=1}^{n} u_k \xrightarrow[n\to\infty]{w} 0$.

Every subspace of a space with property (u) also has property (u).

Theorem [Tzafriri (1972)]

Every subspace of an order continuous Banach lattice has property (u).

Hence, all the spaces in the Theorem c_0 have property (u).

Another class of spaces with property (u) is given by:

Theorem [Gilles + D. Li (1989)]

Every Banach space M-ideal in its bidual has property (u).

Another class of spaces with property (u) is given by:

Theorem [Gilles + D. Li (1989)]

Every Banach space M-ideal in its bidual has property (u).

A Banach space X is M-ideal of its bidual if the natural decomposition $X^{***} = X^* \oplus X^\perp$ is an ℓ_1 decomposition.

About Question 1: M-ideals

Another class of spaces with property (u) is given by:

Theorem [Gilles + D. Li (1989)]

Every Banach space M-ideal in its bidual has property (u).

A Banach space X is M-ideal of its bidual if the natural decomposition $X^{***} = X^* \oplus X^{\perp}$ is an ℓ_1 decomposition.

One has:

Theorem [D. Werner (1989) and Gilles + D. Li (1990)]

If X is a predual of ℓ_1 and is isomorphic to a space M-ideal of its bidual, then X is isomorphic to c_0.

Question 1

Hence the question:

Question [Gilles (1989)]

If X is a predual of ℓ_1 with property (u), is X isomorphic to c_0?

Question 1, particular case: Cyclic Banach spaces

A Banach space X is said to be cyclic if there is a complete Boolean algebra $\mathcal{B}$ of projections and $x_0 \in X$ such that $X = \overline{\mathrm{span}}\,\{Px_0\,;\ P \in \mathcal{B}\}$.

A Banach space X is said to be cyclic if there is a complete Boolean algebra $\mathcal{B}$ of projections and $x_0 \in X$ such that $X = \overline{\mathrm{span}}\,\{Px_0\,;\ P \in \mathcal{B}\}$.

Theorem [Tzafriri (1972)]

1) Every cyclic Banach space has property (u).

Question 1, particular case: Cyclic Banach spaces

A Banach space X is said to be cyclic if there is a complete Boolean algebra $\mathcal{B}$ of projections and $x_0 \in X$ such that $X = \overline{\mathrm{span}}\,\{Px_0\,;\ P \in \mathcal{B}\}$.

Theorem [Tzafriri (1972)]

1) Every cyclic Banach space has property (u).
2) Every order bounded Banach lattice is cyclic.

Question 1, particular case: Cyclic Banach spaces

A Banach space X is said to be cyclic if there is a complete Boolean algebra $\mathcal{B}$ of projections and $x_0 \in X$ such that $X = \overline{\text{span}}\,\{Px_0\,;\; P \in \mathcal{B}\}$.

Theorem [Tzafriri (1972)]

1) Every cyclic Banach space has property (u).
2) Every order bounded Banach lattice is cyclic.

So

Sub-question

Is every cyclic predual of ℓ_1 isomorphic to c_0?

A Banach space X has the hereditarily Dunford-Pettis property (HDPP) if all its subspaces Y have the Dunford-Pettis property (every w-compact $T\colon Y \to Z$ maps wealky convergent sequences into norm convergent ones). Grothendieck showed that c_0 has the (HDPP).

About Question 1: Hereditarily Dunford-Pettis property

A Banach space X has the hereditarily Dunford-Pettis property (HDPP) if all its subspaces Y have the Dunford-Pettis property (every w-compact $T\colon Y \to Z$ maps wealky convergent sequences into norm convergent ones). Grothendieck showed that c_0 has the (HDPP).

Theorem

1) [Cembranos (1987)] A Banach space X has (HDPP) iff it has property (S): every normalized weakly null sequence has a c_0 subsequence.

About Question 1: Hereditarily Dunford-Pettis property

A Banach space X has the hereditarily Dunford-Pettis property (HDPP) if all its subspaces Y have the Dunford-Pettis property (every w-compact $T\colon Y \to Z$ maps wealky convergent sequences into norm convergent ones). Grothendieck showed that c_0 has the (HDPP).

Theorem

1) [Cembranos (1987)] A Banach space X has (HDPP) iff it has property (S): every normalized weakly null sequence has a c_0 subsequence.
2) [Knaust + Odell (1989)] Every Banach space with property (S) has property (u).

About Question 1: Hereditarily Dunford-Pettis property

A Banach space X has the hereditarily Dunford-Pettis property (HDPP) if all its subspaces Y have the Dunford-Pettis property (every w-compact $T\colon Y \to Z$ maps wealky convergent sequences into norm convergent ones). Grothendieck showed that c_0 has the (HDPP).

Theorem

1) [Cembranos (1987)] A Banach space X has (HDPP) iff it has property (S): every normalized weakly null sequence has a c_0 subsequence.
2) [Knaust + Odell (1989)] Every Banach space with property (S) has property (u).

In 1993, Knaust introduced a property, called (FS) (I don't give its definition, but just said that if X has (FS) and X^* is separable, then X has (S)), and proved:

About Question 1: Hereditarily Dunford-Pettis property

A Banach space X has the hereditarily Dunford-Pettis property (HDPP) if all its subspaces Y have the Dunford-Pettis property (every w-compact $T\colon Y \to Z$ maps wealky convergent sequences into norm convergent ones). Grothendieck showed that c_0 has the (HDPP).

Theorem

1) [Cembranos (1987)] A Banach space X has (HDPP) iff it has property (S): every normalized weakly null sequence has a c_0 subsequence.
2) [Knaust + Odell (1989)] Every Banach space with property (S) has property (u).

In 1993, Knaust introduced a property, called (FS) (I don't give its definition, but just said that if X has (FS) and X^* is separable, then X has (S)), and proved:

Theorem [Knaust (1993)]

Every predual of ℓ_1 with property (FS) is isomorphic to c_0.

So:

Sub-question

Is every predual of ℓ_1 with (HDPP) isomorphic to c_0?

The end

I cannot finish without state the following:

The end

I cannot finish without state the following:

Theorem [Gilles+ N. Kalton + P. D. Saphar (1993)]

Let X be an isomorphic predual of ℓ_1 with property (u). Let $d = \operatorname{dist}(X^*, \ell_1)$ and K the constant of property (u), then if

$$Kd^2 < d + 2\,,$$

X is isomorphic to c_0.

The end

I cannot finish without state the following:

Theorem [Gilles+ N. Kalton + P. D. Saphar (1993)]

Let X be an isomorphic predual of ℓ_1 with property (u). Let $d = \operatorname{dist}(X^*, \ell_1)$ and K the constant of property (u), then if

$$Kd^2 < d + 2\,,$$

X is isomorphic to c_0.

MARVELLOUS !

The end

I cannot finish without state the following:

Theorem [Gilles+ N. Kalton + P. D. Saphar (1993)]

Let X be an isomorphic predual of ℓ_1 with property (u). Let $d = \operatorname{dist}(X^*, \ell_1)$ and K the constant of property (u), then if

$$Kd^2 < d + 2\,,$$

X is isomorphic to c_0.

MARVELLOUS !

With such a result the answer to Question 1 cannot be other than:

"yes"!